The Impedance of a Driven String

A guitar string that is excited purely by the vibrations of one (or both) of end supports - *i.e.* the bridge or the nut, is known as a *driven* string. Mechanical energy is input to the string vibration(s) from the vibrations of one (or both) of the end supports.

In order to discuss this phenomenon, we first consider some simplified physical situations. We will also use complex notation to discuss the mathematics associated with this phenomena - see *e.g.* the preceding lecture notes on Fourier analysis for a discussion on complex notation.

Suppose we have an *infinitely* long string, with tension T, and apply a sinusoidally time-varying, complex transverse force of the form $F(t) = |F|e^{i\omega t} = |F|[\cos(\omega t)+i \sin(\omega t)]$ to the end of the string at x = 0, which is a *free* (i.e. not fixed) end, as an *approximation* to *e.g.* a vibrating bridge or nut of a guitar, driving the string. The magnitude of this complex driving force is |F|, a *real* quantity (i.e. a simple constant). Since the string is infinitely long, we do *not* create transverse *standing* waves by driving the string from one end. Instead, transverse *travelling* waves are created by driving the free end of the string in this manner. Thus, the transverse displacement at a point x along the driven string, at time, *t* is given by:

$$y(x,t) = y_o e^{i(\omega t - kx)}$$

Mathematically, this represents a travelling wave moving to the *right* (i.e. increasing x) as time, *t* increases. Note also that the transverse displacement amplitude, y_o is *complex*, i.e. $y_o = |y_o|e^{i\delta} = |y_o|[\cos \delta + i \sin \delta]$ where $|y_o|$ is the *magnitude* of the complex amplitude, y_o and δ is the phase angle, *relative* to the (complex) transverse driving force, F(t). Note also that we have the usual relations, the longitudinal wave speed, $v_x = \omega/k$, with $\omega = 2\pi f$ and $k = 2\pi/\lambda$.

At the driven end of the string, located at *x* = 0, since this end of the string is assumed to be ideally free, there can be no *net* force acting on the string. Therefore the transverse driving force, *F* must balance the transverse component of the tension, T_y, i.e. $F(t) = -T\sin \theta(t)$, which for small amplitude vibrations, corresponding to small-argument Taylor series expansions of $\sin \theta(t) \cong \tan \theta(t) \cong \theta(t) \cong \partial y(x=0, t)/\partial x$ at x = 0, thus the driving force, $F(t) \cong -T(\partial y(x=0, t)/\partial x)$ at x = 0. The slope, $\partial y(x=0, t)/\partial x$ at x = 0 is $\partial y(x=0, t)/\partial x = \partial/\partial x\{y(x, t)\}|_{x=0} = \partial/\partial x\{y_o e^{i(\omega t-kx)}\}|_{x=0} = -ik\, y_o e^{i\omega t} = -ik\, y(x=0, t)$, and hence $F(t) = |F|e^{i\omega t} \cong +ikT\, y(x=0,t) = ikT\, y_o e^{i\omega t} = ikT\, |y_o|e^{i\delta}e^{i\omega t} = ikT\, |y_o|e^{i(\omega t+\delta)}$.

Thus, we see that $|y_o| = |F|e^{-i\delta}/ikT = -i|F|e^{-i\delta}/kT$. However, note that *magnitudes* of *complex* quantities a.) *must* be purely real (i.e. they cannot have any i's in their mathematical expressions)

$$y(x,t) = y_o e^{i(\omega t-kx)} = |\, y_o\,| e^{i\delta} e^{i(\omega t-kx)} = \frac{-i\,|\,F\,|}{kT} e^{i(\omega t-kx)} = -i\frac{|\,F\,|}{kT} e^{i(\omega t-kx)}$$

and b.) non-zero magnitudes of complex quantities *must* be positive. Therefore, the only way that the magnitude, $|y_o|$ can be purely real *and* positive is if $e^{-i\delta}/i = -i\, e^{-i\delta} = -i\, [\cos \delta - i \sin \delta] = -i \cos \delta + (i*i) \sin \delta = -i \cos \delta - \sin \delta = +1$, or $e^{-i\delta} = e^{+i\pi/2} = \cos(\pi/2) + i \sin(\pi/2) = i$. This requires the phase angle, $\delta = -90^o = -\pi/2$. Thus, the complex transverse displacement:

Note that at x = 0, the complex displacement $y(x=0, t) = |F|/kT\, e^{i\omega t}$ is 90^o out of phase with the complex driving force, $F(t) = |F|e^{i\omega t}$.