or $\lambda_n < 2L/n$ for a quasi-fixed/quasi-free end support of mass M located at x = L. A *shorter* wavelength, λ_n implies a *higher* frequency, f_n associated with vibrational modes of a string with one quasi-fixed/quasi-free end support of mass M, in comparison to the vibrational modes associated with rigid/fixed ($M = \infty$) end supports at both ends of the string.

Note also that because of this non-linear relationship (cot(kL) = (M/m)kL), the frequency of the lowest mode (n = 1) is raised *more* from its ($M = \infty$) fixed end-fixed end support value than that of the higher modes (n > 1). Because of this, the frequencies of the higher (n > 1) harmonics will not be precisely at integer multiples of the fundamental (n = 1) - i.e. $f_n \neq nf_1$ for n = 2, 3, 4, ...!!

The complex input impedance of the string at this quasi-fixed/quasi-free end support located at x = L is given by:

$$Z^{input} \equiv \frac{F(t)}{u_y(x=L,t)} = \frac{2ikTy_{oR}\cos(kL)e^{i\omega t}}{+2\omega y_{oR}\sin(kL)e^{i\omega t}} = i\left(\frac{T}{v_x}\right)\cot(kL) = iM\omega$$

From the above transcendental equation, cot(kL) = (M/m)kL. Thus, the complex input impedance is

$$Z^{input} = i\left(\frac{T}{v_x}\right)\cot(kL) = i\left(\frac{T}{v_x}\right)\left(\frac{M}{m}\right)kL = iM\omega$$

Note that this "mass-like" impedance associated with the quasi-free end support at x = L is again purely imaginary, and as stated earlier, positive imaginary. The last term(s) on the right hand side of this relation may seem odd at first, but they aren't. Removing common factors, we have $[T/(mv_x)]kL = \omega$, but since $m = \mu L$, this is $[(T/\mu)/v_x]k = \omega$, and since $(T/\mu) = v_x^2$, this is $v_x k = \omega$, and since $\omega/k = v_x$, then we simply find that 1 = 1!

The power transferred from the quasi-free end mass-like end support located at x = L to the string (or from the string to the quasi-free, mass-like end support, if time is reversed), is given by:

$$P(t) = \frac{\partial E(t)}{\partial t} = F(t)u_y^*(x=L,t) = \frac{|F|^2}{(Z^{input})^*} = 4iM\omega^3\,|y_{oR}|^2\,\sin^2(kL)$$

Again, the power here is purely imaginary - the energy transferred from the vibrating string to this mass-like end support (or vice versa) is (eventually) returned back from the end support of mass, M to the string (or vice versa). We have not (yet) put in the physics of energy flowing from the vibrating string, passing thru such and exciting a quasi-free end support, and then flowing on to other portions of the guitar (or vice versa, for the time-reversed situation)!

Damping Effects in Vibrating Strings

As we learned in the previous lecture notes on waves, a vibrating string has energy associated with it. This energy is imparted to the string at the instant it is e.g. plucked, say at time t = 0 sec. We all know that if we wait long enough, the string vibrations will slowly die out; eventually the string ceases to vibrate altogether. Thus, the energy that was initally imparted to the string by plucking it is *dissipated*. In fact, all of the initial energy associated with the vibrating string ultimately all winds up as heat, somewhere. Dissipation of the string's energy is also known as a form of *damping* of the vibrating string. In general, whenever damping processes exist,

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