or:

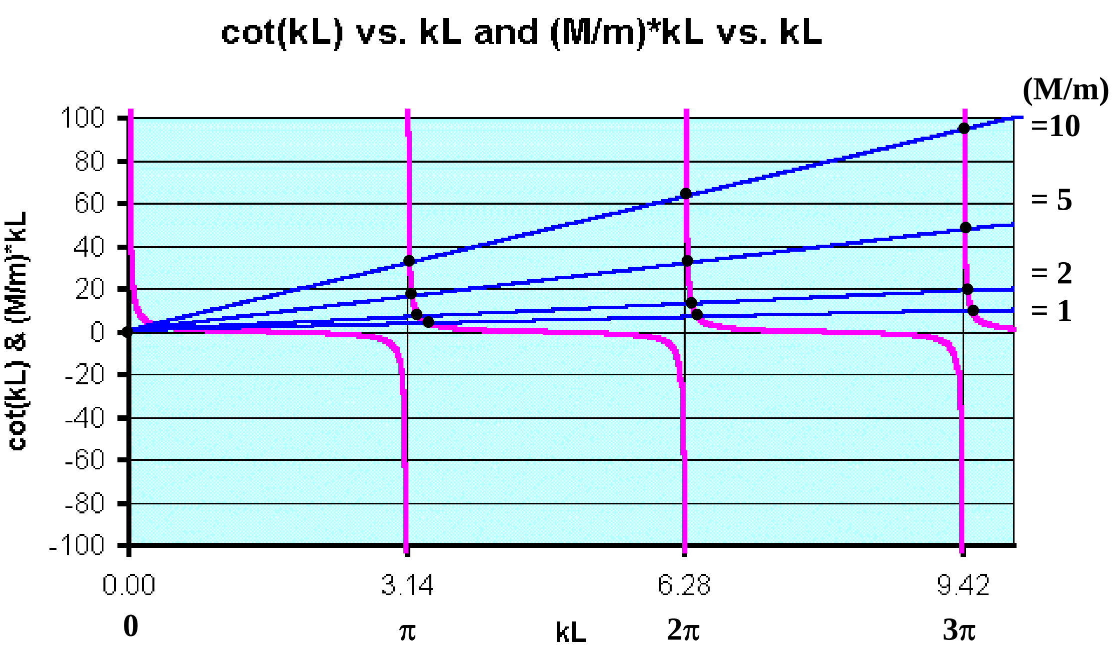
$$kT\cos(kL) = M\omega^2\sin(kL)$$

or:

$$\cot(kL) = \frac{\cos(kL)}{\sin(kL)} = \frac{M\omega^2}{kT} = \frac{kM}{T}\left(\frac{\omega}{k}\right)^2 = \frac{kM}{T}v_x^2 = \frac{kM}{\mu} = \left(\frac{M}{m}\right)kL$$

where we used the relations $v_x = \omega/k$, $v_x^2 = T/\mu$ and $m = \mu L$ = total mass of the string of length, L. The relation cot(kL) = (M/m)kL is a non-linear equation, known as a so-called *transcendental equation* - because it *transcends* known *analytic* mathematical methods for solving this equation. Before the age of computers, solving such equations could only be done in a reasonable amount of time by using graphical techniques!

In the figure below, the magenta curves are the graphs of cot(kL) vs. kL. The dark blue straight lines are the (M/m)kL vs. kL relations, for values of the slope of each straight line, (M/m) = 1, 2, 5 and 10, respectively.



The kL values where each of the dark blue straight-line relations (M/m)kL vs. kL *intersect* with each of the magenta cot(kL) vs. kL curves (shown by black dots) thus determine the vibrational modes of the string with one fixed end support at x = 0 and one quasi-fixed/quasi-free end support of mass, M located at x = L. For example, if the ratio of end support mass to the total string mass is (M/m) = 10, then $k_1L \cong \pi$, $k_2L \cong 2\pi$, $k_3L \cong 3\pi$, etc., or $k_nL \cong n\pi$, hence $k_n \cong n\pi/L$, n = 1, 2, 3, 4, ... (In reality, M >> m on a real guitar), except for the case of slide/bottleneck guitar, if the slide/bottleneck is *e.g.* made of thin glass or plastic. Recall that for fixed end-fixed end boundary conditions on the string (equivalent to M = ∞, and hence (M/m) = ∞), that $k_n = n\pi/L$, n = 1, 2, 3, 4, ... Thus, we see from the above figure, that for finite mass, M of the end support located at x = L, that in fact $k_n > n\pi/L$, and therefore since $k_n = 2\pi/\lambda_n$, then $2\pi/\lambda_n > n\pi/L$,