Now if the transverse waves on the string have small amplitude, this means that the angles, θ and θ' are quite small. Then in this regime, the small-angle approximation holds, such that $\sin\theta \cong \tan\theta \cong \theta$, and $\sin\theta' \cong \tan\theta' \cong \theta'$. But $\tan\theta \cong |\partial y(x+\Delta x, t=0)/\partial x|$ and $\tan\theta' \cong |\partial y(x, t=0)/\partial x|$. Thus, the *magnitude* of the *slope* of the string, $|\partial y(x, t=0)/\partial x|$ is small everywhere on the string for small amplitudes. In other words, for all x, $0 \le x \le L$, where L is the total length of the string, then $|\partial y(x, t=0)/\partial x| << 1$. Then for the snapshot of the string at $t = 0$:

$$F_y = ma_y = T_y + T'_y = |T|\sin\theta - |T'|\sin\theta'$$
$$\cong |T|\,\partial y(x,t=0)/\partial x\,(x+\Delta x) - |T|\,\partial y(x,t=0)/\partial x\,(x)$$
$$= |T|\,[\partial y(x,t=0)/\partial x\,(x+\Delta x) - \partial y(x,t=0)/\partial x\,(x)]$$

This relation holds for *any* time t, not just for $t = 0$. The quantity in the square brackets:

$$[\partial y(x,t=0)/\partial x\,(x+\Delta x) - \partial y(x,t=0)/\partial x\,(x)] = \Delta x\,\partial/\partial x\,(\partial y(x,t)/\partial x) = \Delta x\,(\partial^2 y(x,t)/\partial x^2)$$

by the definition of a double derivative. Thus, $F_y = ma_y \cong |T|\,\Delta x\,(\partial^2 y(x,t)/\partial x^2)$.

If the string has mass per unit length, $\mu = M/L$ (*kg*/*m*), where M = total mass of the string (in *kilograms*) and L = total length of the string (in *meters*), then, for small amplitude waves on the string (where $\Delta y << \Delta x$) the infinitesimal string segment, of length $\Delta L = (\Delta x^2 + \Delta y^2)^{1/2} \cong \Delta x$ has a mass $m \cong \mu\,\Delta x$ (*kg*).

Now the (transverse) acceleration in the y-direction, is $a_y(x, t)$ at the point x, for a given time, t, which by the definition of an acceleration, is the change in the velocity per unit change in time, is given by:

$$a_y(x, t) = \partial u_y(x,t)/\partial t = \partial/\partial t(\partial y(x,t)/\partial t) = \partial^2 y(x,t)/\partial t^2.$$

Then since $F_y = ma_y \cong |T|\,\Delta x\,(\partial^2 y(x,t)/\partial x^2)$ and $F_y = ma_y = \mu\,\Delta x\,\partial^2 y(x,t)/\partial t^2$, we see that:

$$|T|\,\partial^2 y(x,t)/\partial x^2 = \mu\,\partial^2 y(x,t)/\partial t^2$$

Now, it turns out that the *ratio* of the magnitude of the tension, $|T|$ to the mass per unit length of the string, μ is actually the square of the *longitudinal* speed of propagation, $|v_x|^2$ of the waves on the string, *i.e*.

$$|v_x|^2 = |T|/\mu \quad \text{or} \quad |v_x| = (|T|/\mu)^{1/2}$$

Thus, the wave equation for propagation of transverse waves on a string is given by:

$$\partial^2 y(x,t)/\partial x^2 = 1/|v_x|^2\;\;\partial^2 y(x,t)/\partial t^2$$

One can prove that right- and/or left-moving traveling waves, with transverse displacements $y(x, t) = f(x \mp v_x t)$ satisfy the above wave equation, by explicitly carrying out the differentiation on both sides of the wave equation, using the chain-rule of differentiation. We leave this as an exercise for the interested reader.