However, recall that for the n^{th} normal mode of vibration, the wavenumber, $k_n = 2\pi/\lambda_n$ and $\lambda_n = 2L/n$, n = 1, 2, 3, 4, ... *etc*. Thus, $(2k_nL) = (k_n2L) = [(2\pi/\lambda_n)*2L] = [(2\pi n/2L)*2L] = 2\pi n$. But $\sin(2\pi n) = 0$ for n = 1, 2, 3, 4, ... *etc*. Thus, this integral = $L/2$. Hence, the kinetic energy of the standing wave associated with the n^{th} normal mode of vibration of the entire string, of length, L as a function of time, t is

$$KE_n(t) = \tfrac{1}{4}\mu L\omega_n^2 C_n^2 \sin^2(\omega_n t) = \tfrac{1}{4} m\omega_n^2 C_n^2 \sin^2(\omega_n t)$$

Note that the last term on the right hand side of the above equation used the relation for the mass per unit length of the string, $\mu = m/L$.

The slope, $\partial y_n(x,t)/\partial x$ of the the n^{th} normal mode of vibration as a function of position, x and time, t is given by:

$$\frac{\partial y_n(x,t)}{\partial x} = \frac{\partial}{\partial x}\left[C_n \cos(\omega_n t)\sin(k_n x)\right] = C_n \cos(\omega_n t)\frac{\partial}{\partial x}\left[\sin(k_n x)\right] = k_n C_n \cos(\omega_n t)\cos(k_n x)$$

The potential energy of the standing wave associated with the n^{th} normal mode of vibration of the entire string, of length, L as a function of time, t is given by

$$PE_n(t) = \tfrac{1}{2}T\int_{x=0}^{x=L} dx\left(\frac{\partial y(x,t)}{\partial x}\right)^2 = \tfrac{1}{2}T\ \ k_n^2 \cos^2(\omega_n t)\int_{x=0}^{x=L} dx\cos(k_n x)$$

Now the integral

$$\int_{x=0}^{x=L} dx\cos^2(k_n x) = \left[\frac{x}{2} + \frac{\sin(2k_n x)}{4k_n}\right]_{x=0}^{x=L} = \frac{L}{2} + \frac{\sin(2k_n L)}{4k_n}$$

Thus, from the above, we know that this integral = $L/2$. Hence the potential energy of the standing wave associated with the the n^{th} normal mode of vibration of the entire string, of length L, as a function of time, t is

$$PE_n(t) = \tfrac{1}{4}T\ \ L\ k_n^2 C_n^2 \cos^2(\omega_n t)$$

Now, since the longitudinal wave speed, $|v_x| = \omega_n/k_n$, or $k_n = \omega_n/|v_x|$, then

$$PE_n(t) = \tfrac{1}{4}T\ \ L\ k_n^2 C_n^2 \cos^2(\omega_n t) = \tfrac{1}{4}T\ \ L\frac{\omega_n^2}{v_x^2} C_n^2 \cos^2(\omega_n t)$$

However, we also know that $v_x^2 = T/\mu$. Thus, the potential energy of the standing wave associated with the n^{th} normal mode of vibration of the entire string, of length, L as a function of time, t is

$$PE_n(t) = \tfrac{1}{4}\mu L\ \omega_n^2 C_n^2 \cos^2(\omega_n t) = \tfrac{1}{4} m\omega_n^2 C_n^2 \cos^2(\omega_n t)$$

The *total* energy, $E_n(t)$ (in *Joules*) of the standing wave associated with the n^{th} normal mode of vibration of the entire string, of length, L as a function of time, t is

$$E_n(t) = KE_n(t) + PE_n(t) = \tfrac{1}{4} m\omega_n^2 C_n^2 \sin^2(\omega_n t) + \tfrac{1}{4} m\omega_n^2 C_n^2 \cos^2(\omega_n t) = \tfrac{1}{4} m\omega_n^2 C_n^2$$