where we have *also* used the approximation to the Taylor series expansion, $1/(1-\varepsilon) \approx \varepsilon$ for $\varepsilon \approx 0$, in the above formula.

Putting this all together, the potential energy, $\Delta PE(x,t)$ of an infinitesimal string segment of length, Δx for small amplitude string vibrations, is given by:

$$\Delta PE(x,t) = \tfrac{1}{2} T \Delta x \left(\frac{\partial y(x,t)}{\partial x} \right)^2$$

The potential energy, $\Delta KE(x,t)$ associated with a vibrating string segment is a maximum when the magnitude of the slope, |slope| = $|\partial y(x,t)/\partial x|$ of the string segment is a maximum. For sinusoidal-type transverse waves on a string, this occurs when the transverse displacement, $y(x,t)$ is a maximum.

We can again obtain the total potential energy associated with the entire vibrating string at a given instant in time, t by summing up (i.e. adding) all of the individual potential energy contributions associated with each infinitesimal string segment, over the entire length of the string:

$$PE(t) = \sum_{n=1}^{N} \Delta PE(x,t) = \tfrac{1}{2} T \sum_{n=1}^{N} \Delta x \left(\frac{\partial y(x,t)}{\partial x} \right)^2$$

In the limit that the lengths of individual string segments, Δx become truly infinitesimal, then $\Delta x \to dx$, and the above summation formally becomes an integration over the length of the string, say from $x = 0$ to $x = L$:

$$PE(t) = \tfrac{1}{2} T \int_{x=0}^{x=L} dx \left(\frac{\partial y(x,t)}{\partial x} \right)^2$$

As we have seen above, a standing wave in the n^{th} normal mode of vibration has a transverse displacement as a function of position, x and time, t of:

$$y_n(x,t) = C_n \cos(\omega_n t) \sin(k_n x)$$

The transverse velocity, $u_{yn}(x,t)$ of the n^{th} normal mode of vibration as a function of position, x and time, t is given by:

$$u_{yn}(x,t) = \frac{dy(x,t)}{dt} = \frac{d}{dt}\left[C_n \cos(\omega_n t)\sin(k_n x)\right] = C_n \frac{d}{dt}\left[\cos(\omega_n t)\right]\sin(k_n x) = -\omega_n C_n \sin(\omega_n t)\sin(k_n x)$$

The kinetic energy of the standing wave associated with the n^{th} normal mode of vibration of the entire string, of length, L as a function of time, t is given by:

$$KE_n(t) = \tfrac{1}{2} \mu \int_{x=0}^{x=L} dx\ u_{yn}^2(x,t) = \tfrac{1}{2} \mu \omega_n^2 C_n^2 \sin^2(\omega_n t) \int_{x=0}^{x=L} dx \sin^2(k_n x)$$

Now the integral $\int_{x=0}^{x=L} dx \sin^2(k_n x) = \left[\frac{x}{2} - \frac{\sin(2k_n x)}{4k_n} \right]_{x=0}^{x=L} = \frac{L}{2} - \frac{\sin(2k_n L)}{4k_n}$

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