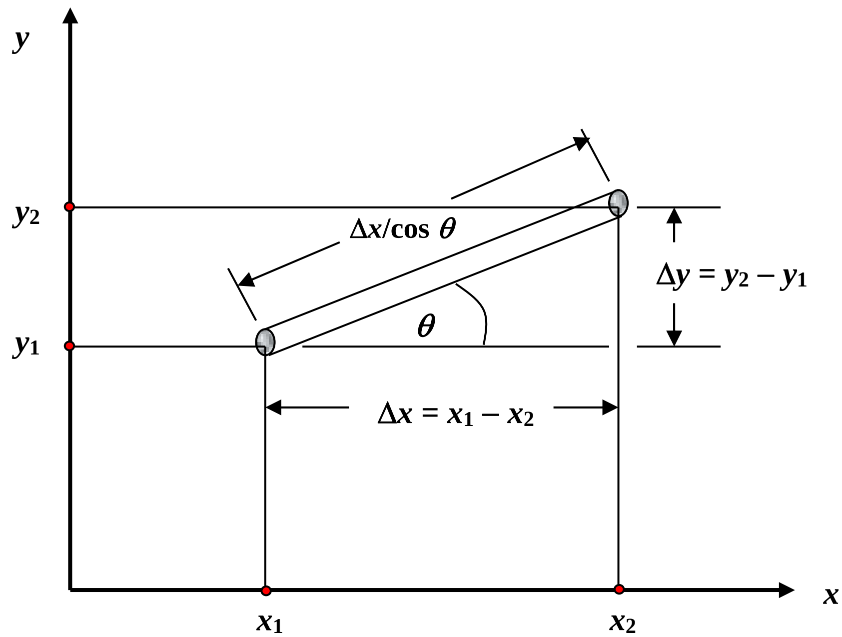

The elongation, ΔL of this infinitesimal string segment in changing from its equilibrium position, $y(x,t) = 0$ to its transversely displaced position, $y(x,t) \neq 0$ is given by

$$\Delta L(x,t) = \frac{\Delta x}{\cos\theta(x,t)} - \Delta x = \Delta x\left(\frac{1}{\cos\theta(x,t)} - 1\right)$$

Note that the mass, $\Delta m = \mu\Delta x$ associated with the (now stretched out) infinitesimal string segment does *not* change in the stretching process.

For small amplitude vibrations the angle, $\theta(x,t)$ is very small. Because of this, we can approximate $\cos\theta$ by the first two leading terms of its Taylor series expansion:

$$\cos\theta = \sum_{n=0}^{n=\infty} \frac{(-1)^n}{2n!}\theta^{2n} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ... \approx 1 - \frac{\theta^2}{2!} = 1 - \frac{\theta^2}{2}$$

where $n! = n(n-1)(n-2)...1$, and $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 12$, *etc*. Now because the angle, θ is small, then $\tan\theta \approx \theta$, but $\tan\theta = \partial y/\partial x$ = slope of the infinitesimal string segment. (The small-angle relation(s): $\tan\theta \approx \theta$, and also $\sin\theta \approx \theta$ are again consequences of truncating the Taylor series expansions for $\tan\theta$ and $\sin\theta$ to the first terms in these expansions.) Thus, for small amplitudes, $\theta(x,t) \approx \partial y(x,t)/\partial x$ = slope of the infinitesimal string segment.

Thus, for small amplitudes, the elongation, $\Delta L(x,t)$ of the string segment is given by:

$$\Delta L(x,t) = \frac{\Delta x}{1 - \frac{1}{2}\left(\frac{\partial y(x,t)}{\partial x}\right)^2 + ...} - \Delta x \approx \tfrac{1}{2}\Delta x\left(\frac{\partial y(x,t)}{\partial x}\right)^2$$