

This can be seen by taking the Fourier transform of a finite-length {time duration Δt_0 } pure-tone/single frequency { $f = f_0$ } **time domain** sinusoidal signal $p(t) = p_0 \sin \omega_0 t$ to the **frequency domain** $p(f)$:

If $p(t) = p_0 \sin \omega_0 t = p_0 \sin 2\pi f_0 t$ for $|t| \leq \frac{1}{2} \Delta t_0$, and: $p(t) = 0$ for $|t| > \frac{1}{2} \Delta t_0$, and defining the {rectangular} **window function** $w(t) = 1$ ($= 0$) for $|t| \leq \frac{1}{2} \Delta t_0$ ($|t| > \frac{1}{2} \Delta t_0$), respectively, then:

$$p(f) = \int_{t=-\infty}^{t=+\infty} p(t) \cdot \sin \omega t \, dt = p_0 \int_{t=-\infty}^{t=+\infty} w(t) \cdot \sin \omega_0 t \cdot \sin \omega t \, dt = p_0 \int_{t=-\frac{1}{2}\Delta t_0}^{t=+\frac{1}{2}\Delta t_0} \sin \omega_0 t \cdot \sin \omega t \, dt$$

Using the trigonometric identity $\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$:

$$p(f) = \frac{1}{2} p_0 \int_{t=-\frac{1}{2}\Delta t_0}^{t=+\frac{1}{2}\Delta t_0} [\cos(\omega - \omega_0)t - \cos(\omega + \omega_0)t] \, dt = \frac{1}{2} p_0 \Delta t_0 \left\{ \frac{\sin\left[\frac{1}{2}(\omega - \omega_0)\Delta t_0\right]}{\left[\frac{1}{2}(\omega - \omega_0)\Delta t_0\right]} - \frac{\sin\left[\frac{1}{2}(\omega + \omega_0)\Delta t_0\right]}{\left[\frac{1}{2}(\omega + \omega_0)\Delta t_0\right]} \right\}$$

The **sinc function** $\text{sinc}(x) \equiv \frac{\sin x}{x}$ {n.b. $\text{sinc}(0) = 1$ }, hence we can write $p(f)$ as:

$$p(f) = \frac{1}{2} p_0 \Delta t_0 \left\{ \text{sinc}\left[\frac{1}{2}(\omega - \omega_0)\Delta t_0\right] - \text{sinc}\left[\frac{1}{2}(\omega + \omega_0)\Delta t_0\right] \right\}$$

The **power spectral density functions** $S_{pp}(f) \propto |p(f)|^2$ (a **frequency domain** quantity) for infinite length and finite length sine-wave signals are shown below:

