Now suppose that a ***small*** amount of damping is present in the system. Mathematically this is represented by $\gamma \equiv (b/2M) < \omega$, hence $\sqrt{\gamma^2 - \omega^2} = i\sqrt{\omega^2 - \gamma^2}$, with purely real $\sqrt{\omega^2 - \gamma^2} > 0$. We define $\omega'^2 \equiv \omega^2 - \gamma^2$ or equivalently $\omega' \equiv \sqrt{\omega^2 - \gamma^2} > 0$. Thus, for ***under-damped*** 1-D harmonic motion, with $0 < \gamma \equiv (b/2M) < \omega$, we see that $\zeta = -\gamma \pm \sqrt{\gamma^2 - \omega^2} = -\gamma \pm i\sqrt{\omega^2 - \gamma^2} = -\gamma \pm i\omega'$, and thus the *physical* solution for ***under-damped*** 1-D harmonic motion, for our ***initial*** conditions is given by: $\boxed{x(t) = x_o e^{\zeta t} = x_o e^{-\gamma t}\cos(\omega' t) = \tfrac{1}{2} x_o e^{-\gamma t}\left(e^{i\omega' t} + e^{-i\omega' t}\right)}$ where the ***damping constant*** $0 < \gamma \equiv (b/2M) < \omega$ and $\boxed{0 < \omega' \equiv \sqrt{\omega^2 - \gamma^2} < \omega}$. The motion is ***exponentially*** damped as time increases, with ***damping time constant*** $\boxed{\tau_d \equiv 1/\gamma = (2M/b)}$ (*seconds*), where the ***envelope*** of the 1-D oscillation falls to $1/e = e^{-1} \simeq 0.3679$ of its initial value at time $\boxed{t = \tau_d \equiv 1/\gamma = (2M/b)}$ (*seconds*), as shown in the figure below:

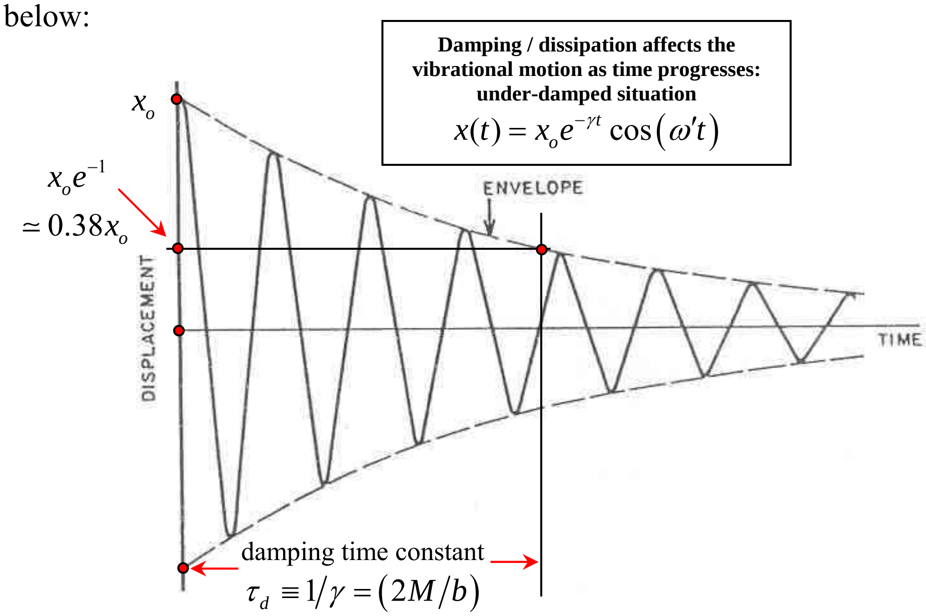

FIG. 3. Graph of displacement versus time for a damped vibration.

Dissipative processes/friction tends to ***lower*** the frequency of oscillation of a vibrating system, as can be seen from the relation $\boxed{0 < \omega' = \sqrt{\omega^2 - \gamma^2} < \omega}$. Small damping corresponds to a slight decrease in the oscillation frequency from its "natural" un-damped value of $\omega = \sqrt{k/M}$.

If we now imagine slowly increasing the damping to "heavy" damping – eventually there will be no oscillation(s) at all! When $\gamma = \omega$, the system is said to be ***critically damped***, and $\zeta = -\gamma$, and the corresponding ***critically-damped*** motion is a purely-decaying exponential with time: $\boxed{x(t) = x_o e^{-\gamma t}}$. When $\gamma > \omega$, the system is said to be ***over-damped***, and $\sqrt{\gamma^2 - \omega^2} > 0$. Here, $\zeta = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$, but the ***physical*** solution is: $\zeta = -\gamma + \sqrt{\gamma^2 - \omega^2} = -\gamma\left(1 - \sqrt{1 - (\omega/\gamma)^2}\right)$.

The ***over-damped*** motion is again a decaying exponential with time: $\boxed{x(t) = x_o e^{\zeta t} = x_o e^{-\gamma\left(1 - \sqrt{1 - (\omega/\gamma)^2}\right)t}}$.

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