Note that $P.E.(t)$, $K.E.(t)$ and $E_{tot}(t)$ are always ≥ 0 (*i.e.* never negative)!!!

Note further that energy/energies are ***additive***, ***scalar*** quantities.

A ***real*** vibrating spring – mass system suffers from various energy loss mechanisms:

- * friction – the mass M slides on surface, mass M also slides through viscous air
- * spring also dissipates energy internally each time it is flexed (another type of friction)

* Thus, the motion of a *real* mass on a *real* spring is damped by frictional processes.
* The original/initial energy, $E_{tot}(t) = E_o$ = constant is dissipated by frictional processes.
* The initial energy E_o ultimately winds up as heat (another form of energy) - thus the mass, spring, horizontal surface and the air all heat up with time…

Mathematically, we can represent the effect(s) of frictional damping associated with a 1-D simple harmonic oscillator as a velocity-dependent (and hence time-dependent) force $F_d(t)$ acting horizontally on the mass M, which opposes the motion, which, for the initial conditions of our problem, this damping force is given by: $\boxed{F_d(t) = +bv(t)}$ where b is a positive constant, known as the ***viscous damping coefficient***, with *SI* units of *kg*/*sec*.

Then since $\boxed{v(t) = \dfrac{dx(t)}{dt} \equiv \dot{x}(t)}$, the equation of motion for the damped 1-D simple harmonic oscillator becomes: $\boxed{M\dfrac{d^2x(t)}{dt^2} + b\dfrac{dx(t)}{dt} + kx(t) = 0}$ or: $\boxed{M\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0}$

We can rewrite this differential equation as: $\boxed{\ddot{x}(t) + (b/M)\dot{x}(t) + (k/M)x(t) = 0}$ and defining: the ***damping constant*** $\boxed{\gamma \equiv (b/2M) > 0}$ and $\boxed{\omega^2 \equiv (k/M) = (2\pi)^2 f^2}$, then our linear, homogeneous 2nd-order differential equation can also be written as: $\boxed{\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega^2 x(t) = 0}$.
The general solution to this differential equation is of the form: $\boxed{x(t) = x_o e^{\zeta t}}$.

Explicitly carrying out the time-differentiation we obtain: $\boxed{\zeta^2 x(t) + 2\gamma\zeta x(t) + \omega^2 x(t) = 0}$ or: $\boxed{\zeta^2 + 2\gamma\zeta + \omega^2 = 0}$ which in turn is a quadratic equation in ζ, the solution for which has two roots: $\boxed{\zeta = \dfrac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega^2}}$.

When ***no*** damping is present ($\gamma = 0$), then: $\boxed{\gamma \equiv (b/2M) = 0}$, and thus: $\boxed{\zeta = \pm\sqrt{-\omega^2}}$.

Defining $i \equiv \sqrt{-1}$, then for $\gamma = 0$ we see that $\boxed{\zeta = \pm i\omega}$.
Next, we use Euler's complex relations for cosine and sine functions: $\boxed{\cos\omega t = \tfrac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)}$ and $\boxed{\sin\omega t = \tfrac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)}$. For the no-damping situation, we already know (from above) what the solution ***must*** be for the 1-D harmonic motion, with our initial conditions:
$\boxed{x(t) = x_o\cos(\omega t) = \tfrac{1}{2}x_o\left(e^{i\omega t} + e^{-i\omega t}\right)}$.