Discrete Fourier Transforms:

In an experimental/laboratory situation *e.g.* using modern data acquisition hardware such as a digital oscilloscope/digital recorder, or a dedicated waveform acquisition system, where ***time domain*** signals $\tilde{f}(t)$ are ***digitized*** at a constant sampling rate of f_s (samples/second) {corresponding to a sampling time interval $\Delta t_s = 1/f_s$} their ***frequency domain*** counterparts $\tilde{g}(\omega)$ can be obtained using so-called ***discretize***d Fast-Fourier Transform {*FFT*} techniques.

For ***discretized*** complex ***time domain*** functions consisting of a uniformly time-sampled sequence of *N* ***complex*** numbers $\tilde{g}_n(t_n)$, the ***discrete*** Fourier transform of complex ***time domain*** $\tilde{g}_n(t_n)$ to the ***frequency domain*** is: $\tilde{g}_k(\omega_k) = \sum_{n=0}^{N-1} \tilde{g}_n(t_n) e^{-(2\pi i/N)kn}$, with $k = 0, 1, 2, \ldots N-1$.

The ***inverse*** of the ***discrete*** Fourier transform of complex ***frequency domain*** $\tilde{g}_k(\omega_k)$ to the ***time domain*** is: $\tilde{g}_n(t_n) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{g}_k(\omega_k) e^{+(2\pi i/N)kn}$. The $e^{(2\pi i/N)}$ are known as the ***primitive N^{th} roots of unity***.

For the remainder of the discussion(s) ***here***, we will work with ***continuous*** complex functions $\tilde{g}(t)$ and. We leave it to the interested reader to transcribe the following results to the discretized versions, if needed, or, one can simply consult a good text book on digital signal processing…

Convolution:

We can ***convolute*** a complex ***time domain*** function $\tilde{f}(t)$ with another $\tilde{g}(t)$ by carrying out the mathematical operation of ***convolution***:

$$\boxed{\tilde{h}_{f\otimes g}(t) \equiv \tilde{f}(t) \otimes \tilde{g}(t) \equiv \int_{-\infty}^{+\infty} \tilde{f}(\tau)\cdot\tilde{g}(t-\tau)\,d\tau}$$

The $\otimes$ symbol denotes the ***convolution*** operation. *n.b.* $\tilde{h}(t)$ has units of $\tilde{f}(t)\cdot\tilde{g}(t)\cdot sec$.

The Fourier transform the above relation is: $\tilde{h}_{f\otimes g}(\omega) = \mathcal{F}\left\{\tilde{h}_{f\otimes g}(t)\right\} \equiv \int_{-\infty}^{+\infty} \tilde{h}_{f\otimes g}(t) e^{-i\omega t} dt$, the ***frequency domain*** representation of complex ***time domain convolution***. It can be shown that:

$$\boxed{\tilde{h}_{f\otimes g}(\omega) = \mathcal{F}\left\{\tilde{h}_{f\otimes g}(t)\right\} = \mathcal{F}\left\{\tilde{f}(t)\otimes\tilde{g}(t)\right\} = \mathcal{F}\left\{\tilde{f}(t)\right\}\cdot\mathcal{F}\left\{\tilde{g}(t)\right\} = \tilde{f}(\omega)\cdot\tilde{g}(\omega)}$$

The $\cdot$ symbol denotes simple ***multiplication***. *n.b.* $\tilde{h}_{f\otimes g}(\omega)$ has physical units of $\tilde{f}(\omega)\cdot\tilde{g}(\omega)$.

We see that ***convolution*** of two complex ***time domain*** functions $\tilde{f}(t)\otimes\tilde{g}(t)$ is equivalent to simple ***multiplication*** of their ***frequency domain*** counterparts, $\tilde{f}(\omega)\cdot\tilde{g}(\omega)$!