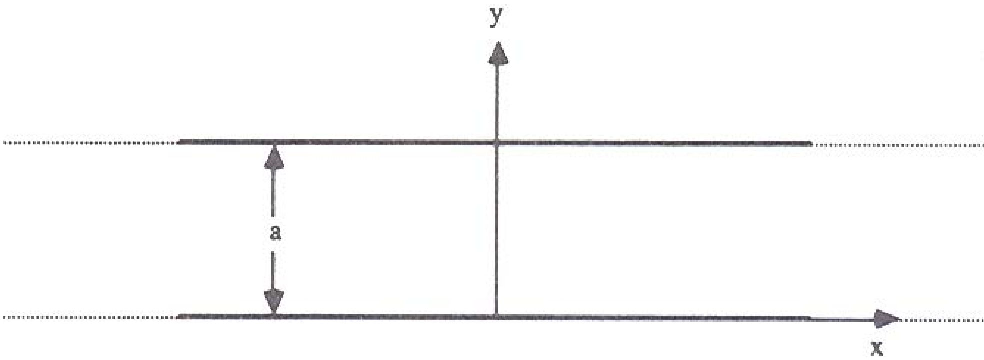
Example # 7: The Uniform Planar Rigid-Walled 2-D Duct:

The final example we wish to discuss is that of sound propagation of monochromatic waves in an infinitely-long uniform planar duct consisting of two infinite, parallel, perfectly reflecting and rigid walls separated by a perpendicular distance *a*, as shown in the figure below:



The wave equation for the complex scalar over-pressure field is:

$$\nabla^2 \tilde{p}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \tilde{p}(\vec{r},t)}{\partial t^2} = 0$$

The sound propagation direction is in the $+\hat{x}$-direction; note that the sound waves are not constrained in the *z*-direction ($+\hat{z}$ points out of the page), whereas sound waves ***are*** constrained in the *y*-direction, being allowed only in the region between the two infinite, parallel walls: $0 \le y \le a$. Thus, this problem is only a 2-D problem in (x, y) rectangular coordinates.

The gradient $\vec{\nabla}$ and Laplacian ∇^2 operators in 2-D Cartesian/rectangular coordinates are:

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} \quad \text{and:} \quad \nabla^2 \equiv \vec{\nabla}\bullet\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The 2-D wave equation for the complex ***time-domain*** scalar pressure field is thus:

$$\frac{\partial^2 \tilde{p}(x,y,t)}{\partial x^2} + \frac{\partial^2 \tilde{p}(x,y,t)}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 \tilde{p}(x,y,t)}{\partial t^2} = 0$$

We seek $+\hat{x}$-propagating wave product-type solutions of the general form:

$$\tilde{p}(x,y,t) = \tilde{X}(x)\tilde{Y}(y)\tilde{T}(t) = Ae^{-ik_x x}e^{\mp ik_y y}e^{i\omega t}$$

The homogeneous wave equation is separable in these variables; the resulting ***characteristic equation*** for the wavenumber *k* is: $k^2 = k_x^2 + k_y^2$, with accompanying ***dispersion relation*** $k^2 = \omega^2/c^2$. {The details of this separation-of-variables technique for the 2-D wave equation are given in the P406 Lecture Notes on "Mathematical Musical Physics of the Wave Equation" *p.* 12-13}.