Since the observer/listener position is on-axis at $\vec{r} = \vec{z} = z\hat{z}$, and the ***on-axis*** complex ***time-domain*** over-pressure $\tilde{p}(r = z, t)$ has axial symmetry (*i.e.* no φ-dependence), the gradient operator $\vec{\nabla} = \partial/\partial r\,\hat{r} \Rightarrow \partial/\partial z\,\hat{z}$, and (after some derivative-taking and some algebra), the ***on-axis*** complex ***time-domain*** particle velocity is:

$$
\begin{aligned}
\tilde{u}_z(r = z, t) &= u_o\left[\left\{1 - e^{-ik\left(\sqrt{z^2+a^2}-z\right)}\right\} - \left\{\frac{z}{\sqrt{z^2+a^2}} - 1\right\} e^{-ik\left(\sqrt{z^2+a^2}-z\right)}\right] \cdot e^{i(\omega t - kz)} \\
&= u_o\left[1 - \frac{1}{\sqrt{1+(a/z)^2}} e^{-ikz\left(\sqrt{1+(a/z)^2}-1\right)}\right] \cdot e^{i(\omega t - kz)} \\
&= u_o\left[e^{-ikz} - \frac{1}{\sqrt{1+(a/z)^2}} e^{-ikz\sqrt{1+(a/z)^2}}\right] \cdot e^{i\omega t} = \tilde{u}_z(r = z, \omega) \cdot e^{i\omega t}
\end{aligned}
$$

The real and imaginary components of the ***on-axis frequency-domain*** complex particle velocity are:

$$
u_{z_r}(r = z, \omega) = \operatorname{Re}\left\{\tilde{u}_z(r = z, \omega)\right\} = u_o\left[\cos kz - \frac{1}{\sqrt{1+(a/z)^2}} \cos kz\sqrt{1+(a/z)^2}\right]
$$

and:

$$
u_{z_i}(r = z, \omega) = \operatorname{Im}\left\{\tilde{u}_z(r = z, \omega)\right\} = -u_o\left[\sin kz - \frac{1}{\sqrt{1+(a/z)^2}} \sin kz\sqrt{1+(a/z)^2}\right]
$$

The ***on-axis*** complex ***specific*** longitudinal acoustic impedance, using $z_o \equiv \rho_o c = p_o/u_o$ is:

$$
\begin{aligned}
\tilde{z}_{a_z}(r = z, \omega) &= \frac{\tilde{p}(r = z, t)}{\tilde{u}_z(r = z, t)} = \frac{\tilde{p}(r = z, \omega) \cdot \cancel{e^{i\omega t}}}{\tilde{u}_z(r = z, \omega) \cdot \cancel{e^{i\omega t}}} = \frac{\tilde{p}(r = z, \omega)}{\tilde{u}_z(r = z, \omega)} \\
&= \frac{p_o\left[1 - e^{-ikz\left(\sqrt{1+(a/z)^2}-1\right)}\right]}{u_o\left[1 - \frac{1}{\sqrt{1+(a/z)^2}} e^{-ikz\left(\sqrt{1+(a/z)^2}-1\right)}\right]} = z_o \frac{\left[1 - e^{-ikz\left(\sqrt{1+(a/z)^2}-1\right)}\right]}{\left[1 - \frac{1}{\sqrt{1+(a/z)^2}} e^{-ikz\left(\sqrt{1+(a/z)^2}-1\right)}\right]}
\end{aligned}
$$