Thus, the dot product $\vec{\rho}\bullet\vec{r} = (\rho\cos\varphi\hat{x} + \rho\sin\varphi\hat{y})\bullet(r\sin\theta\hat{x} + r\cos\theta\hat{z}) = \rho r\sin\theta\cos\varphi$.

Hence: $r'^2 = r^2 - 2\vec{\rho}\bullet\vec{r} + \rho^2 = r^2 - 2\rho r\sin\theta\cos\varphi + \rho^2$,

Or: $r' = \sqrt{r'^2} = \sqrt{r^2 - 2\vec{\rho}\bullet\vec{r} + \rho^2} = \sqrt{r^2 - 2\rho r\sin\theta\cos\varphi + \rho^2}$

Next, if we carry out the integration of the above integral over the surface area of the plane circular piston of radius *a* using *e.g.* the method of annular strips, the area element can be written as $dS = \rho d\varphi \cdot d\rho$. The surface integral becomes:

$$\begin{aligned}\tilde{p}(r,t) &= i\frac{\rho_o\omega u_o}{2\pi}\int\frac{1}{r'}e^{i(\omega t - kr')}dS = i\frac{\rho_o\omega u_o}{2\pi}\int_{\varphi=0}^{\varphi=2\pi}\int_{\rho=0}^{\rho=a}\frac{1}{r'}e^{i(\omega t-kr')}\rho d\varphi d\rho \\ &= i\frac{\rho_o\omega u_o}{2\pi}e^{i\omega t}\int_{\varphi=0}^{\varphi=2\pi}\int_{\rho=0}^{\rho=a}\frac{1}{\sqrt{r^2-2\rho r\sin\theta\cos\varphi+\rho^2}}e^{-ik\sqrt{r^2-2\rho r\sin\theta\cos\varphi+\rho^2}}\rho d\varphi d\rho\end{aligned}$$

This "***off-axis***" integral is ***not*** easy to carry out – analytically – it is a so-called ***elliptic integral*** (of the "würst" kind). It ***can*** be done *e.g.* ***numerically***, on a computer. If the observer/listener position is located somewhere on the $\hat{z}$-axis, *i.e.* $\vec{r} = z\hat{z}$ {when $\theta = 0$}, the "***on-axis***" integral is ***much*** easier to carry out – analytically, because the dot product $\vec{\rho}\bullet\vec{r}$ vanishes, the "***on-axis***" problem has axial symmetry {*i.e.* has rotational invariance about the $\hat{z}$-axis}:

$$\begin{aligned}\tilde{p}(r=z,t) &= i\frac{\rho_o\omega u_o}{2\pi}e^{i\omega t}\int_{\varphi=0}^{\varphi=2\pi}\int_{\rho=0}^{\rho=a}\frac{1}{\sqrt{z^2+\rho^2}}e^{-ik\sqrt{z^2+\rho^2}}\rho d\varphi d\rho \\ &= i\,\cancel{2\pi}\frac{\rho_o\omega u_o}{\cancel{2\pi}}e^{i\omega t}\int_{\rho=0}^{\rho=a}\frac{\rho}{\sqrt{z^2+\rho^2}}e^{-ik\sqrt{z^2+\rho^2}}d\rho\end{aligned}$$

Note that the ***integrand*** of the ρ-integral is a ***perfect differential***:

$$\frac{\rho}{\sqrt{z^2+\rho^2}}e^{-ik\sqrt{z^2+\rho^2}} = -\frac{d}{d\rho}\left[\frac{e^{-ik\sqrt{z^2+\rho^2}}}{ik}\right]$$

Noting also that $c = \omega/k$, $z_o \equiv \rho_o c$ and $p_o = z_o u_o = \rho_o c u_o$, the ***on-axis time-domain*** complex over-pressure is thus:

$$\begin{aligned}\tilde{p}(r=z,t) &= -i\rho_o\omega u_o e^{i\omega t}\int_{\rho=0}^{\rho=a}\frac{d}{d\rho}\left[\frac{e^{-ik\sqrt{z^2+\rho^2}}}{ik}\right]d\rho = -i\rho_o\omega u_o e^{i\omega t}\int_{\rho=0}^{\rho=a}d\left[\frac{e^{-ik\sqrt{z^2+\rho^2}}}{ik}\right] \\ &= -\frac{\cancel{i}\rho_o\omega u_o}{\cancel{i}k}e^{i\omega t}\left[e^{-ik\sqrt{z^2+\rho^2}}\right]_{\rho=0}^{\rho=a} = +\underbrace{\rho_o c}_{\equiv z_o}u_o\left[1-e^{-ik\left(\sqrt{z^2+a^2}-z\right)}\right]\cdot e^{i(\omega t-kz)} \\ &= p_o\left[1-e^{-ikz\left(\sqrt{1+(a/z)^2}-1\right)}\right]\cdot e^{i(\omega t-kz)} = p_o\left[e^{-ikz}-e^{-ikz\sqrt{1+(a/z)^2}}\right]\cdot e^{i\omega t} = \tilde{p}(r=z,\omega)\cdot e^{i\omega t}\end{aligned}$$