Examples of Complex Sound Fields (Continued):

Example # 3: Point Monopole Sound Source – Spherical Waves Propagating in "Free Air":

Note: there exists ***no*** electromagnetic analog for this acoustic example, due to the manifest ***vectorial*** nature of the *EM* field – which is mediated at the microscopic level by the spin-1 photon. So-called ***electric monopole*** {$E(0)$} and/or ***magnetic monopole*** {$M(0)$} *EM* ***radiation*** associated *e.g.* with a spherically-symmetric, radially oscillating electric charge distribution $\rho_e(\vec{r},t) = q_o\delta^3(\vec{r})e^{i\omega t}$ and/or magnetic charge distribution $\rho_m(\vec{r},t) = g_o\delta^3(\vec{r})e^{i\omega t}$ ***cannot*** occur.

Imagine a spherically-symmetric, point sound source located at the origin of coordinates $\vec{r} = 0$ that isotropically emits monochromatic ***spherical*** acoustic waves into "***free air***". The 3-D wave equation describing the behavior of the ***instantaneous/physical*** {*i.e.* purely ***real time-domain***} over-pressure $p(\vec{r},t)$ at the space-time point $(\vec{r},t)$ is an ***inhomogeneous***, linear 2nd-order differential equation:

$$\nabla^2 p(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 p(\vec{r},t)}{\partial t^2} = -4\pi B_o\delta^3(\vec{r})\cos\omega t$$

The gradient $\vec{\nabla}$ and Laplacian $\nabla^2 = \vec{\nabla}\bullet\vec{\nabla}$ operators in 3-D spherical-polar (r,θ,φ) coordinates are:

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}$$

and:

$$\nabla^2 \equiv \vec{\nabla}\bullet\vec{\nabla} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

The RHS of this 3-D wave equation is originates from $\nabla^2(1/r) = -4\pi\delta^3(\vec{r})$, thus $-4\pi B_o\delta^3(\vec{r})\cos\omega t$ describes the point sound source located at the origin $\vec{r} = 0$, radiating sound isotropically into 4π steradians. The function $\delta^3(\vec{r})$ is known as {Dirac's} 3-D ***delta function***, which has many intriguing mathematical properties, one of which is that the 3-D delta function $\delta^3(\vec{r})$ has an (***infinite***!) spike at the origin $\vec{r} = 0$ and is $\equiv 0$ elsewhere. Thus, we can equivalently write $\delta^3(\vec{r}) = \delta^3(\vec{r} - 0)$. Note that in spherical-polar coordinates (r,θ,φ), that $\delta^3(\vec{r}) = \frac{1}{4\pi r^2}\delta(r)$ where $\delta(r) = \delta(r-0)$ is the 1-D delta function in the radial (r) direction only. If we integrate the 3-D delta function over a volume V containing the origin $\vec{r} = 0$, *e.g.* integrate over ***all*** space:

$$\begin{aligned}\int_V \delta^3(\vec{r})dV &= \int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=\infty}\delta^3(\vec{r})\cdot r^2 dr\sin\theta d\theta d\varphi = \int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=\infty}\frac{1}{4\pi \cancel{r^2}}\delta(r)\cdot\cancel{r^2}dr\sin\theta d\theta d\varphi \\ &= \tfrac{1}{4\pi}\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=\infty}\delta(r)\cdot dr\sin\theta d\theta d\varphi = \frac{1}{2\cdot\cancel{2\pi}}\cancel{2\pi}\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=\infty}\delta(r)\cdot dr\sin\theta d\theta \\ &= \tfrac{1}{2}\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=\infty}\delta(r)\cdot dr\cdot d\cos\theta = \tfrac{1}{2}\int_{u=-1}^{u=1}\int_{r=0}^{r=\infty}\delta(r)\cdot dr\cdot du = \tfrac{1}{\cancel{2}}\cancel{2}\int_{r=0}^{r=\infty}\delta(r)\cdot dr = 1\end{aligned}$$

If the volume V does ***not*** contain the origin $\vec{r} = 0$, then: $\int_V \delta^3(\vec{r})dV = 0$.