If we now take the spatial gradient of both sides of the linearized mass continuity equation, and the time derivative of both sides of the linearized Euler equation, and again use the {linearized} adiabatic relationship between complex overpressure, $\tilde{p}$ and mass density, $\tilde{\rho}(\vec{r},t)=\frac{1}{c^2}\tilde{p}(\vec{r},t)$, we also have the relation: $\nabla\tilde{\rho}(\vec{r},t)=\frac{1}{c^2}\nabla\tilde{p}(\vec{r},t)$, then:

$$\nabla\left(\vec{\nabla}\bullet\vec{\tilde{u}}(\vec{r},t)\right)\simeq-\frac{1}{\rho_o}\frac{\partial\nabla\tilde{\rho}(\vec{r},t)}{\partial t}=-\frac{1}{\rho_o c^2}\frac{\partial\nabla\tilde{p}(\vec{r},t)}{\partial t}\quad\text{and:}\quad\frac{\partial^2\vec{\tilde{u}}(\vec{r},t)}{\partial t^2}\simeq-\frac{1}{\rho_o}\frac{\partial\vec{\nabla}\tilde{p}(\vec{r},t)}{\partial t}$$

Combining these two equations, we obtain:

$$\vec{\nabla}\left(\vec{\nabla}\bullet\vec{\tilde{u}}(\vec{r},t)\right)=-\frac{1}{\rho_o c^2}\frac{\partial\vec{\nabla}\tilde{p}(\vec{r},t)}{\partial t}=\frac{1}{c^2}\frac{\partial^2\vec{\tilde{u}}(\vec{r},t)}{\partial t^2}$$

If the complex vector acoustic particle velocity field is ***irrotational*** (*i.e.* $\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)=0$), then using the vector relation $\nabla^2\vec{\tilde{u}}=\vec{\nabla}\left(\vec{\nabla}\bullet\vec{\tilde{u}}\right)-\vec{\nabla}\times\left(\cancel{\vec{\nabla}\times\vec{\tilde{u}}}\right)=\vec{\nabla}\left(\vec{\nabla}\bullet\vec{\tilde{u}}\right)$, we also obtain the {linearized} wave equation for complex vector particle velocity:

$$\boxed{\nabla^2\vec{\tilde{u}}(\vec{r},t)-\frac{1}{c^2}\frac{\partial^2\vec{\tilde{u}}(\vec{r},t)}{\partial t^2}=0}$$

The Complex Particle Velocity Potential, $\tilde{\Phi}_u(\vec{r},t)$

Since an inviscid (*i.e.* dissipationless) fluid does not support vorticity, *i.e.* $\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)=0$ then since the ***curl*** of the ***gradient*** of any ***arbitrary*** scalar field $f(\vec{r},t)$ is also ***always*** zero, *i.e.* $\vec{\nabla}\times\vec{\nabla}f(\vec{r},t)\equiv 0$, we can write $\vec{\tilde{u}}(\vec{r},t)=\vec{\nabla}\tilde{\Phi}_u(\vec{r},t)$, where $\tilde{\Phi}_u(\vec{r},t)$ is the ***complex particle velocity potential*** associated with $\vec{\tilde{u}}(\vec{r},t)$. Then $\vec{\nabla}\times\vec{\nabla}\tilde{\Phi}_u(\vec{r},t)=0$.

Note that since $\vec{\tilde{u}}(\vec{r},t)$ and the gradient operator $\vec{\nabla}\equiv\partial/\partial x\,\hat{x}+\partial/\partial y\,\hat{y}+\partial/\partial z\,\hat{z}$ {in Cartesian coordinates} have SI units of m/s and m^{-1} respectively, the complex velocity potential $\tilde{\Phi}_u(\vec{r},t)$ has SI units of m^2/s. Physically, note also that lines/contours {and/or 3-D surfaces} of constant $\tilde{\Phi}_u(\vec{r},t)=\tilde{K}=k+i\kappa=constant$ are thus {complex!} "***equipotentials***", which are {everywhere} ***perpendicular*** to the complex particle velocity $\vec{\tilde{u}}(\vec{r},t)$.

Note additionally that $\tilde{\Phi}_u(\vec{r},t)$ with $\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)=\vec{\nabla}\times\vec{\nabla}\tilde{\Phi}_u(\vec{r},t)=0$ is the acoustic analog of the electrostatic potential $\Phi_e(\vec{r})$ associated with the electrostatic field $\vec{E}(\vec{r})\equiv-\vec{\nabla}\tilde{\Phi}_e(\vec{r})$, since in ***electrostatics*** $\vec{\nabla}\times\vec{E}(\vec{r})\equiv-\vec{\nabla}\times\vec{\nabla}\tilde{\Phi}_e(\vec{r})=0$ {whereas in ***electrodynamics***, $\vec{\nabla}\times\vec{E}(\vec{r},t)\equiv-\partial\vec{B}(\vec{r},t)/\partial t\neq 0$ }.