What is a ***Fourier transform***?

For ***continuous*** complex ***time-domain*** functions $\tilde{f}(t)$, the ***Fourier transform*** of the complex ***time-domain*** function $\tilde{f}(t)$ to the complex ***frequency-domain*** is: $\boxed{\tilde{f}(\omega) \equiv \int_{-\infty}^{+\infty} \tilde{f}(t) e^{-i\omega t} dt}$ where $t$ is treated as a {dummy} variable in the integration over {all} time, from $-\infty \leq t \leq +\infty$.

The ***inverse*** ***Fourier transform*** of a ***continuous*** complex ***frequency-domain*** function $\tilde{f}(\omega)$ to the ***time-domain*** is: $\boxed{\tilde{f}(t) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(\omega) e^{+i\omega t} d\omega}$ where $\omega = 2\pi f$ is treated as a {dummy} variable in the integration over {all negative **.and.** positive} angular frequencies: $-\infty \leq \omega \leq +\infty$.

Note also that the factor of $1/2\pi$ appears here pre-multiplying the latter integral over the angular frequency variable $\omega$ because we are using the angular frequency $\omega \equiv 2\pi f$ in the integral rather than the frequency $f$ itself as a {dummy} variable of integration – technically speaking, frequency $f$ (*sec*$^{-1}$) and time $t$ (*seconds*) are ***true*** ***Fourier conjugate variables*** of each other, and ***not*** angular frequency $\omega = 2\pi f$ (***radians***/*sec*$^{-1}$) and time $t$ (*seconds*).

For monochromatic/single-frequency (*aka* harmonic) sound fields the relationship between "generic" complex ***time-domain*** *vs*. complex ***frequency-domain*** quantities is simply given by $\tilde{f}(t) = \tilde{f}(\omega) \cdot e^{i\omega t}$. Thus, *e.g*. the relations between complex ***time-domain*** *vs*. complex ***frequency-domain*** scalar over-pressure and/or 3-D complex vector particle velocity are:

$$\tilde{p}(t) = \tilde{p}(\omega) \cdot e^{i\omega t} = \left|\tilde{p}(\omega)\right| \cdot e^{i\varphi_p(\omega)} \cdot e^{i\omega t}$$

and:

$$\begin{aligned} \vec{\tilde{u}}(t) = \vec{\tilde{u}}(\omega) \cdot e^{i\omega t} &= \left(\tilde{u}_x(\omega)\hat{x} + \tilde{u}_y(\omega)\hat{y} + \tilde{u}_z(\omega)\hat{z}\right) \cdot e^{i\omega t} \\ &= \left(\left|\tilde{u}_x(\omega)\right| \cdot e^{i\varphi_{u_x}(\omega)}\hat{x} + \left|\tilde{u}_y(\omega)\right| \cdot e^{i\varphi_{u_y}(\omega)}\hat{y} + \left|\tilde{u}_z(\omega)\right| \cdot e^{i\varphi_{u_z}(\omega)}\hat{z}\right) \cdot e^{i\omega t} \end{aligned}$$

There are several useful relations associated with Fourier transforms which we list here:

| | Time-Domain: | | Frequency Domain: |
|---|---|---|---|
| Linearity: | $\tilde{h}(t) = a\tilde{f}(t) + b\tilde{g}(t)$ | $\Rightarrow$ | $\tilde{h}(\omega) = a\tilde{f}(\omega) + b\tilde{g}(\omega)$ |
| Translation: | $\tilde{h}(t) = \tilde{f}(t - t_o)$ | $\Rightarrow$ | $\tilde{h}(\omega) = \tilde{f}(\omega) e^{i\omega t_o}$ |
| Modulation: | $\tilde{h}(t) = \tilde{f}(t) e^{i\omega_o t}$ | $\Rightarrow$ | $\tilde{h}(\omega) = \tilde{f}(\omega - \omega_o)$ |
| Scaling: | $\tilde{h}(t) = \tilde{f}(at)$ | $\Rightarrow$ | $\tilde{h}(\omega) = \frac{1}{\lvert a \rvert} \tilde{f}\left(\frac{\omega}{a}\right)$ |
| Conjugation: | $\tilde{h}(t) = \tilde{f}^*(t)$ | $\Rightarrow$ | $\tilde{h}(\omega) = \tilde{f}^*(-\omega)$ |