Euler's Equation for Inviscid Fluid Flow

Euler's equation for *inviscid* (*i.e.* *dissipationless*) fluid flow is a special/limiting case of the more general {non-linear} Navier-Stokes equation – which expresses Newton's 2nd law of motion for {compressible} fluid flow. The N-S eq′n, in the absence of external driving forces is:

$$\tilde{\rho}(\vec{r},t)\frac{D\vec{\tilde{u}}(\vec{r},t)}{Dt} = -\vec{\nabla}\tilde{p}(\vec{r},t) + \left(\tfrac{4}{3}\eta + \eta_B\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{\tilde{u}}(\vec{r},t)\right) - \eta\left(\vec{\nabla}\times\left(\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)\right)\right)$$

The two ***dissipative*** terms on the right-hand side of the Navier-Stokes equation – a non-zero gradient of the divergence of the particle velocity $\vec{\nabla}\left(\vec{\nabla}\cdot\vec{\tilde{u}}(\vec{r},t)\right)$ and the curl of the ***vorticity*** of the particle velocity $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)\right)$ are associated with the ***coefficient of shear viscosity*** of the fluid η, and the ***coefficient of bulk viscosity*** of the fluid η_B, both of which have SI units of Pascal-seconds (*Pa-s*).

The time derivative term on the left-hand side of the Navier-Stokes equation, $\frac{D\vec{\tilde{u}}(\vec{r},t)}{Dt}$ is the complex particle ***acceleration*** associated with an infinitesimal volume element V of fluid {*e.g.* air} centered on the space-time point $(\vec{r},t)$. From dimensional analysis, note that $\tilde{\rho}(\vec{r},t)\frac{D\vec{\tilde{u}}(\vec{r},t)}{Dt}\left(\frac{kg\text{-}m/s^2}{m^3} = \frac{N}{m^3}\right)$ is a force ***density***. The term $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{\tilde{u}}(\vec{r},t)\cdot\vec{\nabla}$ is known as the ***convective*** (or ***substantive*** , *aka* ***material***) derivative, computed from a ***stationary*** observer's reference frame, *e.g.* fixed in the ***laboratory***:

$$\begin{aligned}\frac{D}{Dt} &\equiv \frac{\partial}{\partial t} + \frac{\partial\tilde{x}(\vec{r},t)}{\partial t}\frac{\partial}{\partial x} + \frac{\partial\tilde{y}(\vec{r},t)}{\partial t}\frac{\partial}{\partial y} + \frac{\partial\tilde{z}(\vec{r},t)}{\partial t}\frac{\partial}{\partial z}\\ &= \frac{\partial}{\partial t} + \tilde{u}_x(\vec{r},t)\frac{\partial}{\partial x} + \tilde{u}_y(\vec{r},t)\frac{\partial}{\partial y} + \tilde{u}_z(\vec{r},t)\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \left(\vec{\tilde{u}}(\vec{r},t)\cdot\vec{\nabla}\right)\end{aligned}$$

Euler's equation for inviscid fluid flow is a first-order, linear, homogeneous differential equation, arising from consideration of momentum conservation in a ***lossless/dissipationless*** compressible fluid (liquid or gas), that in the absence of external driving forces describes the relationship between complex pressure $\tilde{p}(\vec{r},t)$ and complex particle velocity $\vec{\tilde{u}}(\vec{r},t)$ in the compressible fluid, of volume mass density $\tilde{\rho}(\vec{r},t)$ $\left(kg/m^3\right)$. Euler's equation for inviscid fluid flow is thus valid for fluids where the ***viscosity*** of the fluid and/or the ***conduction*** of ***heat*** in the fluid are ***both*** zero {or can both be ***approximated*** as being ***negligible***}:

$$\tilde{\rho}(\vec{r},t)\frac{D\vec{\tilde{u}}(\vec{r},t)}{Dt} = \tilde{\rho}(\vec{r},t)\left(\frac{\partial\vec{\tilde{u}}(\vec{r},t)}{\partial t} + \left(\vec{\tilde{u}}(\vec{r},t)\cdot\vec{\nabla}\right)\vec{\tilde{u}}(\vec{r},t)\right) = -\vec{\nabla}\tilde{p}(\vec{r},t)$$