Thus, in the above figure, the phase angle $\Delta\psi(\vec{r},t)$ associated with the overall/resultant amplitude $\tilde{Z}_{tot}(\vec{r},t)$ is $\tan\Delta\psi = \mathrm{Im}\{\tilde{Z}_{tot}\}/\mathrm{Re}\{\tilde{Z}_{tot}\}$ or: $\Delta\psi = \tan^{-1}\left(\mathrm{Im}\{\tilde{Z}_{tot}\}/\mathrm{Re}\{\tilde{Z}_{tot}\}\right)$.

Writing the "zero-of-time redefined" complex amplitude "vectors" $\tilde{Z}_1(\vec{r},t) = A_1(\vec{r},t)e^{i(\omega_1(t)t)}$, $\tilde{Z}_2(\vec{r},t) = A_2(\vec{r},t)e^{i(\omega_2(t)t+\Delta\varphi_{21}(\vec{r},t))}$ and $\tilde{Z}_{tot}(\vec{r},t) = A_1(\vec{r},t)e^{i\omega_1(t)t} + A_2(\vec{r},t)e^{i\omega_2(t)t}e^{i\Delta\varphi_{21}(\vec{r},t)}$ in terms of their respective real (x-) and imaginary (y-) components, it is straightforward to show that:

$$\Delta\psi(\vec{r},t) = \tan^{-1}\left(\frac{\mathrm{Im}\{\tilde{Z}_{tot}(\vec{r},t)\}}{\mathrm{Re}\{\tilde{Z}_{tot}(\vec{r},t)\}}\right) = \tan^{-1}\left(\frac{A_1(\vec{r},t)\sin(\omega_1 t) + A_2(\vec{r},t)\sin(\omega_2 t + \Delta\varphi_{12})}{A_1(\vec{r},t)\cos(\omega_1 t) + A_2(\vec{r},t)\cos(\omega_2 t + \Delta\varphi_{12})}\right)$$

Note that at $t = 0$:

$$\Delta\psi(\vec{r},t=0) = \tan^{-1}\left(\frac{\mathrm{Im}\{\tilde{Z}_{tot}(\vec{r},t=0)\}}{\mathrm{Re}\{\tilde{Z}_{tot}(\vec{r},t=0)\}}\right) = \tan^{-1}\left(\frac{A_2(\vec{r},t=0)\sin\Delta\varphi_{12}}{A_1(\vec{r},t=0) + A_2(\vec{r},t=0)\cos\Delta\varphi_{12}}\right)$$

If the two frequencies are equal to each other, *i.e.* $\omega_1(t) = \omega_2(t) = \omega$, then $\Delta\omega_{12}(t) \equiv (\omega_1(t) - \omega_2(t)) = 0$ and this expression simplifies to:

$$\Delta\psi(\vec{r},t) = \tan^{-1}\left(\frac{\mathrm{Im}\{\tilde{Z}_{tot}(\vec{r},t)\}}{\mathrm{Re}\{\tilde{Z}_{tot}(\vec{r},t)\}}\right) = \tan^{-1}\left(\frac{A_1(\vec{r},t)\sin\omega t + A_2(\vec{r},t)\sin(\omega t + \Delta\varphi_{12})}{A_1(\vec{r},t)\cos\omega t + A_2(\vec{r},t)\cos(\omega t + \Delta\varphi_{12})}\right)$$

At $t = 0$: $\Delta\psi(\vec{r},t=0) = \tan^{-1}\left(\frac{\mathrm{Im}\{\tilde{Z}_{tot}(\vec{r},t=0)\}}{\mathrm{Re}\{\tilde{Z}_{tot}(\vec{r},t=0)\}}\right) = \tan^{-1}\left(\frac{A_2(\vec{r},t=0)\sin\Delta\varphi_{12}}{A_1(\vec{r},t=0) + A_2(\vec{r},t=0)\cos\Delta\varphi_{12}}\right)$

Finally, if <u>additionally</u> the two individual amplitudes are also equal to each other, *i.e.* $A_1(\vec{r},t) = A_2(\vec{r},t) = A(\vec{r},t)$ then:

$$\left|\tilde{Z}_{tot}(\vec{r},t)\right|^2 = 2A^2(1 + \cos\Delta\varphi_{12}) \text{ and:}$$

$$\Delta\psi(\vec{r},t) = \tan^{-1}\left(\frac{\cancel{A}\sin\omega t + \cancel{A}\sin(\omega t + \Delta\varphi_{12})}{\cancel{A}\cos\omega t + \cancel{A}\cos(\omega t + \Delta\varphi_{12})}\right) = \tan^{-1}\left(\frac{\sin\omega t + \sin(\omega t + \Delta\varphi_{12})}{\cos\omega t + \cos(\omega t + \Delta\varphi_{12})}\right)$$

At $t = 0$: $\Delta\psi(\vec{r},t=0) = \tan^{-1}\left(\frac{\sin\Delta\varphi_{12}}{1 + \cos\Delta\varphi_{12}}\right)$