The same mathematical formalism can be used for adding together two *arbitrary* complex periodic time-dependent signals $\tilde{Z}_1(\vec{r},t) = A_1(\vec{r},t)e^{i(\omega_1(t)t+\varphi_1(\vec{r},t))}$ and $\tilde{Z}_2(\vec{r},t) = A_2(\vec{r},t)e^{i(\omega_2(t)t+\varphi_2(\vec{r},t))}$. Note that *here*, the individual amplitudes, frequencies and phases may all be time-dependent. The resultant overall complex amplitude in this case is:

$$\tilde{Z}_{tot}(\vec{r},t) = \tilde{Z}_1(\vec{r},t) + \tilde{Z}_2(\vec{r},t) = A_1(\vec{r},t)e^{i(\omega_1(t)t+\varphi_1(\vec{r},t))} + A_2(\vec{r},t)e^{i(\omega_2(t)t+\varphi_2(\vec{r},t))}$$

Because the zero of time is (always) arbitrary, we are again free to choose/redefine $t = 0$ in such a way as to rotate away one of the two phases – absorbing it as an overall/absolute phase (which is physically unobservable). Since $e^{(x+y)} = e^x \cdot e^y$, the above formula can be rewritten as:

$$\tilde{Z}_{tot}(\vec{r},t) = \tilde{Z}_1(\vec{r},t) + \tilde{Z}_2(\vec{r},t) = A_1(\vec{r},t)e^{i\omega_1(t)t}e^{i\varphi_1(\vec{r},t)} + A_2(\vec{r},t)e^{i\omega_2(t)t}e^{i\varphi_2(\vec{r},t)}$$

Multiplying both sides of this equation by $e^{-i\varphi_1(t)}$:

$$\begin{aligned}\tilde{Z}_{tot}(\vec{r},t)e^{-i\varphi_1(\vec{r},t)} &= A_1(\vec{r},t)e^{i\omega_1(t)t}\,\cancel{e^{i\varphi_1(\vec{r},t)}e^{-i\varphi_1(\vec{r},t)}} + A_2(\vec{r},t)e^{i\omega_2(t)t}e^{i\varphi_2(\vec{r},t)}e^{-i\varphi_1(\vec{r},t)} \\ &= A_1(\vec{r},t)e^{i\omega_1(t)t} + A_2(\vec{r},t)e^{i\omega_2(t)t}e^{i(\varphi_2(\vec{r},t)-\varphi_1(\vec{r},t))}\end{aligned}$$

This shift in overall phase, by an amount $e^{-i\varphi_1(\vec{r},t)}$ is formally equivalent to a redefinition to the zero of time, and also physically corresponds to a (simultaneous) rotation of both of the (mutually-perpendicular) real and imaginary axes in the complex plane by an angle, $\varphi_1(\vec{r},t)$.

The physical meaning of the remaining phase after this redefinition of time/shift in overall phase is a phase *difference* between the second complex amplitude, $\tilde{Z}_2(\vec{r},t)$ relative to the first, $\tilde{Z}_1(\vec{r},t)$. The relative phase difference is $\Delta\varphi_{12}(\vec{r},t) \equiv (\varphi_1(\vec{r},t) - \varphi_2(\vec{r},t))$. Thus, at the (newly) redefined time $t^* = t - \varphi_1(r,t)/\omega_1(t) = 0$ (and then substituting $t^* \Rightarrow t$) the resulting overall, time-redefined amplitude is:

$$\tilde{Z}_{tot}(\vec{r},t) = A_1(\vec{r},t)e^{i\omega_1(t)t} + A_2(\vec{r},t)e^{i\omega_2(t)t}e^{i(\varphi_2(\vec{r},t)-\varphi_1(\vec{r},t))}$$

or:

$$\tilde{Z}_{tot}(\vec{r},t) = A_1(\vec{r},t)e^{i\omega_1(t)t} + A_2(\vec{r},t)e^{i\omega_2(t)t}e^{i\Delta\varphi_{21}(\vec{r},t)}$$

The magnitude of the resulting overall amplitude, $|\tilde{Z}_{tot}(\vec{r},t)|$ can be obtained from (temporarily suppressing the $(\vec{r},t)$-dependence, for clarity's sake):

$$\begin{aligned}|\tilde{Z}_{tot}|^2 &= \tilde{Z}_{tot} \cdot \tilde{Z}^*_{tot} = (\tilde{Z}_1 + \tilde{Z}_2)\cdot(\tilde{Z}_1 + \tilde{Z}_2)^* = (\tilde{Z}_1 + \tilde{Z}_2)\cdot(\tilde{Z}^*_1 + \tilde{Z}^*_2) \\ &= \tilde{Z}_1 \cdot \tilde{Z}^*_1 + \tilde{Z}_1 \cdot \tilde{Z}^*_2 + \tilde{Z}_2 \cdot \tilde{Z}^*_1 + \tilde{Z}_2 \cdot \tilde{Z}^*_2 \\ &= |\tilde{Z}_1|^2 + \tilde{Z}_1 \cdot \tilde{Z}^*_2 + \tilde{Z}_2 \cdot \tilde{Z}^*_1 + |\tilde{Z}_2|^2\end{aligned}$$

Let us now work on simplifying the sum of the two cross terms in the above expression. Since $\tilde{Z}_1(\vec{r},t)$ and $\tilde{Z}_2(\vec{r},t)$ are complex quantities, they can always be written as:

$$\tilde{Z}_1(\vec{r},t) = X_1(\vec{r},t) + iY_1(\vec{r},t) \text{ and } \tilde{Z}_2(\vec{r},t) = X_2(\vec{r},t) + iY_2(\vec{r},t).$$