We can (always) redefine the phase variable φ such that *e.g.* $\varphi \Rightarrow (\omega t + \varphi)$, it can then be seen that:

$$\tilde{Z}(t) = |\tilde{Z}(t)|(\cos(\omega t + \varphi) + i\sin(\omega t + \varphi))$$

with real component: $X(t) = \mathrm{Re}\{\tilde{Z}(t)\} = |\tilde{Z}(t)|\cos(\omega t + \varphi)$

and imaginary component: $Y(t) = \mathrm{Im}\{\tilde{Z}(t)\} = |\tilde{Z}(t)|\sin(\omega t + \varphi)$.

Note that at the zero of time $t = 0$, these relations are identical to the above.

If (for simplicity's sake) we take the phase angle $\varphi = 0$, then: $\tilde{Z}(t) = |\tilde{Z}(t)|(\cos(\omega t) + i\sin(\omega t))$.

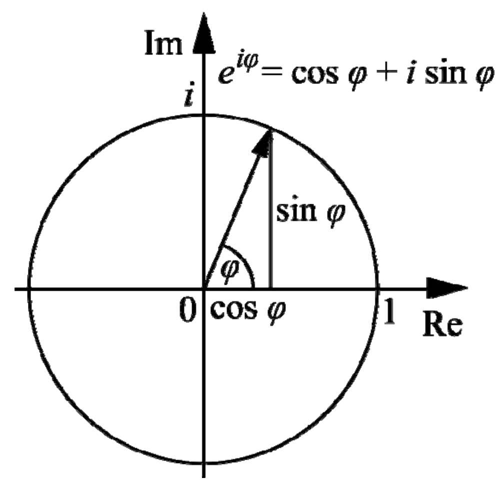
At time $t = 0$, it can be seen that the complex variable $\tilde{Z}(t=0) = X(t=0) = |\tilde{Z}(t=0)|$ is a purely real quantity, lying entirely along the x-axis, since $\tilde{Z}(t=0) = |\tilde{Z}|\cos 0 = |\tilde{Z}(t=0)|$.

As time t progresses, it can be seen that the complex variable $\tilde{Z}(t) = |\tilde{Z}(t)|(\cos(\omega t) + i\sin(\omega t))$ rotates in a ***counter-clockwise*** direction in the complex plane with constant angular frequency $\omega = 2\pi f$ radians/second, where f is the frequency (in cycles/second {cps}, or Hertz {= *Hz*}) completing one revolution in the complex plane every $\tau = 1/f = 2\pi/\omega$ seconds {the variable τ is known as the period of oscillation, or period of vibration}. This rotation of $\tilde{Z}(t)$ in the complex plane can also be seen from the time evolution of the phase:

$$\varphi(t) = \tan^{-1}(Y(t)/X(t)) = \tan^{-1}\left(\cancel{|\tilde{Z}(t)|}\sin\omega t \Big/ \cancel{|\tilde{Z}(t)|}\cos\omega t\right) = \tan^{-1}(\tan\omega t) = \omega t.$$

Complex Exponential Notation:

The famous mathematician-physicist Leonhard Euler showed that for any real number φ, that $e^{i\varphi} = \cos\varphi + i\sin\varphi$. This is known as Euler's formula. Geometrically, the locus of points described by $e^{i\varphi}$ for $0 \le \varphi \le 2\pi$ lie on the unit circle (i.e. radius $|e^{i\varphi}| = 1$) in the complex plane, centered at (0,0) as shown in the figure on the right. Note that if $e^{i\varphi} = \cos\varphi + i\sin\varphi$, then $(e^{i\varphi})^* = e^{-i\varphi} = \cos\varphi - i\sin\varphi$.



We can thus write any "generic" complex quantity $\tilde{Z} = |\tilde{Z}|(\cos\varphi + i\sin\varphi)$ as $\tilde{Z} = |\tilde{Z}|e^{i\varphi}$ and write its complex conjugate $\tilde{Z}^* = |\tilde{Z}|(\cos\varphi - i\sin\varphi)$ as $\tilde{Z}^* = |\tilde{Z}|e^{-i\varphi}$. Note that:

$\tilde{Z}\tilde{Z}^* = (|\tilde{Z}|e^{i\varphi})\cdot(|\tilde{Z}|e^{-i\varphi}) = |\tilde{Z}|^2 e^{i\varphi}\cdot e^{-i\varphi} = |\tilde{Z}|^2 e^{i\varphi - i\varphi} = |\tilde{Z}|^2 e^0 = |\tilde{Z}|^2\cdot 1 = |\tilde{Z}|^2$ Note further that since $e^{i\varphi} = \cos\varphi + i\sin\varphi$ and $e^{-i\varphi} = \cos\varphi - i\sin\varphi$, adding and subtracting these two equations from each other, it is easy to show that $\cos\varphi = \frac{1}{2}(e^{i\varphi} + e^{-i\varphi})$ and $\sin\varphi = \frac{1}{2i}(e^{i\varphi} - e^{-i\varphi})$.