## **Complex Standing Waves**:

Suppose that we linearly superpose (*i.e.* add together) *e.g.* two counter-propagating *scalar* 1-D complex monochromatic traveling plane waves of the same frequency *f* and amplitude *A*, propagating in the $\hat{k}_1 = +\hat{y}$ and $\hat{k}_2 = -\hat{y}$ directions, respectively in a lossless/dispersionless medium. Then $\vec{k}_1 = +k\hat{y} = +(\omega/v)\hat{y} = +(2\pi f/v)\hat{y}$ and $\vec{k}_2 = -k\hat{y} = -(\omega/v)\hat{y} = -(2\pi f/v)\hat{y}$. At the observer's space-time position $(\vec{r},t) = (x,y,z,t)$, the total/resultant wave is:

$$\begin{aligned}\tilde{\psi}_{tot}(\vec{r},t) = \tilde{\psi}_1(\vec{r},t) + \tilde{\psi}_2(\vec{r},t) &= Ae^{i(\omega t - \vec{k}_1\cdot\vec{r} + \varphi_1(\vec{r},t))} + Ae^{i(\omega t - \vec{k}_2\cdot\vec{r} + \varphi_2(\vec{r},t))} \\ &= Ae^{i(\omega t - ky + \varphi_1(\vec{r},t))} + Ae^{i(\omega t + ky + \varphi_2(\vec{r},t))} \\ &= A\left\{e^{i(\omega t - ky + \varphi_1(\vec{r},t))} + e^{i(\omega t + ky + \varphi_2(\vec{r},t))}\right\} \\ &= Ae^{i\omega t}\left\{e^{-i(ky - \varphi_1(\vec{r},t))} + e^{+i(ky + \varphi_2(\vec{r},t))}\right\}\end{aligned}$$

The magnitude (*i.e.* length) of the total/resultant wave is:

$$\begin{aligned}\left|\tilde{\psi}_{tot}(\vec{r},t)\right| &\equiv \sqrt{\tilde{\psi}_{tot}(\vec{r},t)\cdot\tilde{\psi}^*_{tot}(\vec{r},t)} \\ &= \sqrt{A\cancel{e^{i\omega t}}\left\{e^{-i(ky-\varphi_1(\vec{r},t))} + e^{+i(ky+\varphi_2(\vec{r},t))}\right\}\cdot A\cancel{e^{-i\omega t}}\left\{e^{+i(ky-\varphi_1(\vec{r},t))} + e^{-i(ky+\varphi_2(\vec{r},t))}\right\}} \\ &= A\sqrt{\left\{e^{-i(ky-\varphi_1(\vec{r},t))} + e^{+i(ky+\varphi_2(\vec{r},t))}\right\}\cdot\left\{e^{+i(ky-\varphi_1(\vec{r},t))} + e^{-i(ky+\varphi_2(\vec{r},t))}\right\}} \\ &= A\sqrt{1 + e^{+i(2ky+[\varphi_2(\vec{r},t)-\varphi_1(\vec{r},t)])} + e^{-i(2ky+[\varphi_2(\vec{r},t)-\varphi_1(\vec{r},t)])} + 1} \\ &= A\sqrt{2 + e^{+i(2ky+[\varphi_2(\vec{r},t)-\varphi_1(\vec{r},t)])} + e^{-i(2ky+[\varphi_2(\vec{r},t)-\varphi_1(\vec{r},t)])}}\end{aligned}$$

We define: $\Delta\varphi_{21}(\vec{r},t) \equiv \varphi_2(\vec{r},t) - \varphi_1(\vec{r},t)$, thus: $\left|\tilde{\psi}_{tot}(\vec{r},t)\right| = A\sqrt{2 + e^{+i(2ky+\Delta\varphi_{21}(\vec{r},t))} + e^{-i(2ky+\Delta\varphi_{21}(\vec{r},t))}}$

We then define: $\Phi(\vec{r},t) \equiv 2ky + \Delta\varphi_{21}(\vec{r},t)$, thus: $\left|\tilde{\psi}_{tot}(\vec{r},t)\right| = A\sqrt{2 + \underbrace{e^{+i\Phi(\vec{r},t)} + e^{-i\Phi(\vec{r},t)}}_{=2\cos\Phi(\vec{r},t)}}$.

But: $e^{+i\Phi(\vec{r},t)} + e^{-i\Phi(\vec{r},t)} = 2\cos\Phi(\vec{r},t)$, thus we see: $\left|\tilde{\psi}_{tot}(\vec{r},t)\right| = \sqrt{2}A\sqrt{1+\cos\Phi(\vec{r},t)}$.

Thus, we see that when: $\cos\Phi(\vec{r},t) = +1$, *i.e.* when $\Phi(\vec{r},t) = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi ... = \pm n_{even}\pi$, $n_{even}$ = 0, 2, 4, 6, … the total/resultant wave will be maximal (*i.e.* constructive interference of the two counter-propagating traveling waves): $\left|\tilde{\psi}_{tot}(\vec{r},t)\right| = 2A$, but when $\cos\Phi(\vec{r},t) = -1$, *i.e.* when $\Phi(\vec{r},t) = \pm 1\pi, \pm 3\pi, \pm 5\pi ... = \pm n_{odd}\pi$, $n_{odd}$ = 1, 3, 5, 7, … the total/resultant wave will be minimal, (*i.e.* destructive interference of the two counter-propagating traveling waves): $\left|\tilde{\psi}_{tot}(\vec{r},t)\right| = 0$.

If we choose the observer's position *e.g.* to be at $\vec{r} = (x, y=0, z)$ {*i.e.* anywhere in the *x*-*z* plane, at *y* = 0}, then: $\Phi(x,y=0,z,t) \equiv \Delta\varphi_{21}(x,y=0,z,t) = \varphi_2(x,y=0,z,t) - \varphi_1(x,y=0,z,t)$ and we see that: $\cos\Phi(x,y=0,z,t) = \cos\Delta\varphi_{21}(x,y=0,z,t)$, hence the total/resultant plane



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