A Special/Limiting Case – Amplitude Modulation:

Suppose at the observation point $\vec{r}$ in 3-D space that $A_1(\vec{r},t) \gg A_2(\vec{r},t)$ and $f_1 \gg f_2$, then the *exact* expression for the complex total/resultant amplitude:

$$\left|\tilde{Z}_{tot}(\vec{r},t)\right| = \sqrt{\left|\tilde{Z}_{tot}(\vec{r},t)\right|^2} = \sqrt{A_1^2(\vec{r},t) + A_2^2(\vec{r},t) + 2A_1(\vec{r},t)\cdot A_2(\vec{r},t)\cos\left[\Delta\omega_{12}t + \Delta\varphi_{12}(\vec{r},t)\right]}$$

can be *approximated*, neglecting terms of order $m^2 \equiv \left(A_2(\vec{r},t)/A_1(\vec{r},t)\right)^2 \ll 1$ under the radical sign, and noting that for $f_1 \gg f_2$, then $\omega_1 \gg \omega_2$ and hence $\Delta\omega_{12} \equiv (\omega_1 - \omega_2) \cong \omega_1$. For simplicity in this discussion, we set the phase difference $\Delta\varphi_{12}(\vec{r},t) \equiv \varphi_1(\vec{r},t) - \varphi_2(\vec{r},t) = 0$ (its effect is merely to shift the overall beats pattern to the left or right along the time axis). Then:

$$\left|\tilde{Z}_{tot}(\vec{r},t)\right| = A_1(\vec{r},t)\sqrt{1 + \left(A_2(\vec{r},t)/A_1(\vec{r},t)\right)^2 + 2\left(A_2(\vec{r},t)/A_1(\vec{r},t)\right)\cos\left[\Delta\omega_{12}t + \cancel{\Delta\varphi_{12}(\vec{r},t)}\right]}$$

$$= A_1(\vec{r},t)\sqrt{1 + \cancel{m^2} + 2m\cos\left[\Delta\omega_{12}t\right]} \simeq A_1(\vec{r},t)\sqrt{1 + 2m\cos\omega_1 t}$$

Using the Taylor series expansion $\sqrt{1+\varepsilon} \simeq 1 + \frac{1}{2}\varepsilon$ for the case $\varepsilon \equiv 2m\cos\omega_1 t \ll 1$, the magnitude of the total complex amplitude is $\left|\tilde{Z}_{tot}(\vec{r},t)\right| \simeq A_1(\vec{r},t)\left(1 + m\cos\omega_1 t\right)$.

The ratio $m \equiv \left(A_2(\vec{r},t)/A_1(\vec{r},t)\right) \ll 1$ is known as the (amplitude) *modulation depth* associated with the high-frequency *carrier* wave $\tilde{Z}_1(\vec{r},t)$, with amplitude $A_1(\vec{r},t) \gg A_2(\vec{r},t)$ and frequency $f_1 \gg f_2$, *modulated* by the low frequency wave $\tilde{Z}_2(\vec{r},t)$ with amplitude $A_2(\vec{r},t)$ and frequency f_2. This is the underlying principle of how *AM* *radio* works – note that *AM* stands for ***A**mplitude **M**odulation*. In *AM* radio broadcasting, $540\ KHz \lesssim f_1 = f_{carrier} \lesssim 1600\ KHz$ whereas $20Hz \lesssim f_2 = f_{audio} \lesssim 20\ KHz$.

Propagation of Complex Sound Waves in Three Dimensions:

In previous lectures, we have discussed the propagation of purely real sound waves in one dimension, *e.g.* a monochromatic traveling plane wave propagating in the $\pm x$-direction: $\psi(x,t) = A\cos(\omega t \mp kx)$ where A is the amplitude of the wave, the wavenumber $k = 2\pi/\lambda\ (m^{-1})$, the wavelength $\lambda = v/f\ (m)$ and the ***phase*** speed of propagation of the monochromatic traveling wave in the medium is $v_\varphi = f\lambda = \omega/k\ (m/s)$, which in "free air" {*i.e.* "The Great Wide Open"} is also equal to the speed of propagation of energy v_E in that medium.

We can "complexify" the purely real 1-D monochromatic traveling plane wave description(s) $\psi(x,t) = A\cos(\omega t \mp kx)$ to become complex 1-D monochromatic traveling plane waves simply by adding on a purely imaginary term: $iA\sin(\omega t \mp kx)$, *i.e.* complex 1-D monochromatic traveling plane waves in the $\pm x$-direction are mathematically described by:

$$\tilde{\psi}(x,t) = A\left\{\cos(\omega t \mp kx) + i\sin(\omega t \mp kx)\right\} = Ae^{i(\omega t \mp kx)}.$$