Beats Phenomenon

The phenomenon of beats is actually one of the most general cases of wave interference. Suppose at the observation point $\vec{r}$ in 3-D space we linearly superpose (*i.e.* add) together two signals with "zero-of-time re-defined" complex amplitudes $\tilde{Z}_1(\vec{r},t) = A_1(\vec{r},t)e^{i(\omega_1(t)t)}$ and $\tilde{Z}_2(\vec{r},t) = A_2(\vec{r},t)e^{i(\omega_2(t)t+\Delta\varphi_{21}(\vec{r},t))}$, which have similar/comparable frequencies, $\omega_1(t) \sim \omega_2(t)$ with $\Delta\omega_{12}(t) \equiv (\omega_1(t) - \omega_2(t))$ and instantaneous phase of the second signal relative to the first of $\Delta\varphi_{12}(\vec{r},t) \equiv (\varphi_1(\vec{r},t) - \varphi_2(\vec{r},t))$, the total/overall complex amplitude at the observation point $\vec{r}$ in 3-D space is $\tilde{Z}_{tot}(\vec{r},t) = \tilde{Z}_1(\vec{r},t) + \tilde{Z}_2(\vec{r},t) = A_1(\vec{r},t)e^{i\omega_1(t)t} + A_2(\vec{r},t)e^{i\omega_2(t)t}e^{i\Delta\varphi_{21}(\vec{r},t)}$.

Note that at the amplitude level, there is nothing explicitly overt and/or obvious in the above mathematical expression for the overall/total/resultant complex amplitude $\tilde{Z}_{tot}(\vec{r},t)$ that *easily* at-a-glance explains the phenomenon of beats associated with adding together two complex signals that have comparable amplitudes and frequencies.

However, let's consider the (instantaneous) phasor relationship between the two complex amplitudes $\tilde{Z}_1(\vec{r},t) = A_1(\vec{r},t)e^{i(\omega_1(t)t)}$ and $\tilde{Z}_2(\vec{r},t) = A_2(\vec{r},t)e^{i(\omega_2(t)t+\Delta\varphi_{21}(\vec{r},t))}$ respectively. Their relative phase difference at time $t = 0$ is $\Delta\varphi_{12}(\vec{r},t=0) \equiv (\varphi_1(\vec{r},t=0) - \varphi_2(\vec{r},t=0))$; the resultant/total complex amplitude $\tilde{Z}_{tot}(\vec{r},t)$ is shown in the above phasor diagram at time $t = 0$.

From the law of cosines, we showed above that magnitude^2 of the resultant/total complex amplitude at the observation point $\vec{r}$ in 3-D space was:

$$\left|\tilde{Z}_{tot}(\vec{r},t)\right|^2 = A_1^2(\vec{r},t) + A_2^2(\vec{r},t) + 2A_1(\vec{r},t)\cdot A_2(\vec{r},t)\cos\left[\Delta\omega_{12}t + \Delta\varphi_{12}(\vec{r},t)\right]$$

Then:

$$\left|\tilde{Z}_{tot}(\vec{r},t)\right| = \sqrt{\left|\tilde{Z}_{tot}(\vec{r},t)\right|^2} = \sqrt{A_1^2(\vec{r},t) + A_2^2(\vec{r},t) + 2A_1(\vec{r},t)\cdot A_2(\vec{r},t)\cos\left[\Delta\omega_{12}t + \Delta\varphi_{12}(\vec{r},t)\right]}$$

For equal amplitudes: $A_1(\vec{r},t) = A_2(\vec{r},t) = A(\vec{r},t) = A = constant$ and zero relative phase: $\Delta\varphi_{12}(\vec{r},t) \equiv (\varphi_1(\vec{r},t) - \varphi_2(\vec{r},t)) = 0$ $\{i.e.\ \varphi_1(\vec{r},t) = \varphi_2(\vec{r},t)\}$ and constant (*i.e.* time-independent) frequencies ω_2 and ω_1, this expression simplifies to:

$$\left|\tilde{Z}_{tot}(\vec{r},t)\right| = \sqrt{A^2 + A^2 + 2A^2\cos\Delta\omega_{12}t} = \sqrt{2}A\cdot\sqrt{1+\cos\Delta\omega_{12}t}$$

Note that as time t increases, that $0 \le \left|\tilde{Z}_{tot}(\vec{r},t)\right| \le 2$.

The phase $\Delta\psi(\vec{r},t)$ associated with the total amplitude $\tilde{Z}_{tot}(\vec{r},t)$ for this specialized case is:

$$\Delta\psi(\vec{r},t) = \tan^{-1}\left(\frac{\mathrm{Im}\left\{\tilde{Z}_{tot}(\vec{r},t)\right\}}{\mathrm{Re}\left\{\tilde{Z}_{tot}(\vec{r},t)\right\}}\right) = \tan^{-1}\left(\frac{\sin(\omega_1 t) + \sin(\omega_2 t)}{\cos(\omega_1 t) + \cos(\omega_2 t)}\right)$$