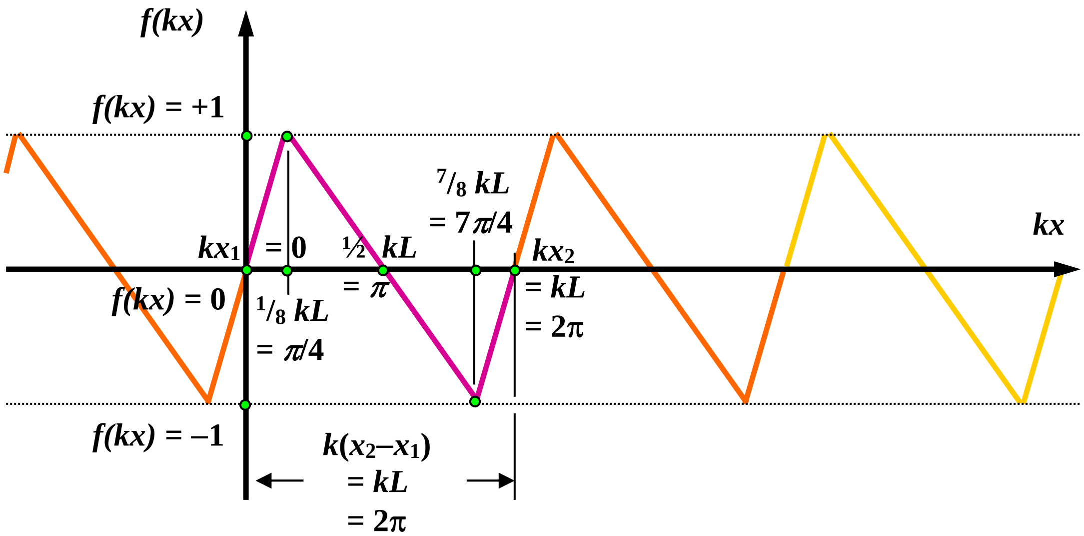
E. Fourier Analysis of a Periodic Sawtooth (Asymmetrical Triangle) Wave

Next, we consider a spatially-periodic bipolar *sawtooth* wave, i.e. an *asymmetrical* bipolar triangle wave of *unit* amplitude, as shown in the figure below:



Mathematically, this *odd*-symmetry waveform, on the "generic" interval $0 \le \theta < 2\pi$ (i.e. one cycle of this waveform) is described as:

$$f(\theta) = f(kx) = +(4/\pi)\theta \qquad \text{for} \qquad 0 \le \theta < \pi/4$$

and:

$$f(\theta) = f(kx) = -(4/3\pi)\theta + 4/3 \quad \text{for} \quad \pi/4 \le \theta < 7\pi/4$$

and:

$$f(\theta) = f(kx) = +(4/\pi)\theta - 8 \quad \text{for} \quad 7\pi/4 \le \theta < 2\pi$$

Where we used the straight line equation, $y = mx + b$ to determine the slopes, m and the intercepts, b associated with each of the three line segments in the above waveform on this θ-interval.

We determine the Fourier coefficients, a_0, a_n and b_n from the inner products:

$$a_0 = \frac{1}{\pi}\langle f(\theta),1\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)d\theta$$

$$= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi/4} f(\theta)d\theta + \int_{\theta=\pi/4}^{\theta=7\pi/4} f(\theta)d\theta + \int_{\theta=7\pi/4}^{\theta=2\pi} f(\theta)d\theta\right]$$

$$= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi/4} (\tfrac{4}{\pi})\theta d\theta + \int_{\theta=\pi/4}^{\theta=7\pi/4} \left((\tfrac{-4}{3\pi})\theta + \tfrac{4}{3}\right)d\theta + \int_{\theta=7\pi/4}^{\theta=2\pi} \left((\tfrac{4}{\pi})\theta - 8\right)d\theta\right]$$

$$= \frac{1}{\pi}\left[(\tfrac{4}{\pi})\tfrac{1}{2}\theta^2 \,\big|_0^{\pi/4} + [(\tfrac{-4}{3\pi})\tfrac{1}{2}\theta^2 + \tfrac{4}{3}\theta]\,\big|_{\pi/4}^{7\pi/4} + [(\tfrac{4}{\pi})\tfrac{1}{2}\theta^2 - 8\theta]\,\big|_{7\pi/4}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\tfrac{1}{\pi}\left[\tfrac{1}{4}\pi^2 - \tfrac{9}{4}\pi^2 + \tfrac{1}{4}\pi^2 + 4\pi^2 - \tfrac{9}{4}\pi^2\right] + \left[3\pi - \pi - 8\pi + 6\pi\right]\right] = 0$$