Note further that this bipolar, 50% duty-cycle square wave has a “d.c. offset”, or “time”-averaged value (averaged over one cycle) of $<f(\theta)> = 0$. Formally, the averaging of a periodic function over one of its cycles, which for the θ-variable, one cycle in theta is $\Delta\theta = \theta_2 - \theta_1 = 2\pi$, is mathematically defined as:

$$\langle f(\theta)\rangle = \frac{1}{\Delta\theta}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)d\theta = \frac{1}{2\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)d\theta$$

For the periodic bipolar, 50% duty-cycle square wave, the θ-averaging of this waveform over one θ-cycle is:

$$\langle f(\theta)\rangle = \frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)d\theta = \frac{1}{2\pi}\left[\int_{\theta=0}^{\theta=\pi} f(\theta)d\theta + \int_{\theta=\pi}^{\theta=2\pi} f(\theta)d\theta\right] = \frac{1}{2\pi}\left[\int_{\theta=0}^{\theta=\pi}(+1)d\theta + \int_{\theta=\pi}^{\theta=2\pi}(-1)d\theta\right]$$
$$= \frac{1}{2\pi}\left[\int_{\theta=0}^{\theta=\pi} d\theta - \int_{\theta=\pi}^{\theta=2\pi} d\theta\right] = \frac{1}{2\pi}\left[\theta\,\Big|_{\theta=0}^{\theta=\pi} - \theta\,\Big|_{\theta=\pi}^{\theta=2\pi}\right] = \frac{1}{2\pi}[\pi-\pi] = \frac{\pi}{2\pi}[1-1] = \frac{1}{2}[1-1] = 0$$

QED.

We now obtain the Fourier coefficients a_0, a_n and b_n by taking the following inner products:

$$a_0 = \frac{1}{\pi}\langle f(\theta),1\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)d\theta = \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi} f(\theta)d\theta + \int_{\theta=\pi}^{\theta=2\pi} f(\theta)d\theta\right]$$
$$= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi} d\theta - \int_{\theta=\pi}^{\theta=2\pi} d\theta\right] = \frac{1}{\pi}[\pi-\pi] = \frac{\pi}{\pi}[1-1] = \frac{1}{1}[1-1] = [1-1] = 0$$

Thus, if the reader compares the inner product for determining a_0 with that for obtaining the θ-averaged value of $f(\theta)$, i.e. $<f(\theta)>$, one sees that:

$$\langle f(\theta)\rangle = \frac{a_0}{2}$$

which is just what we expected!

Now, recall that here, since $\theta = \omega t$, the “generic” variable, $\theta_n = n\omega t = n\theta$. Thus:

$$a_n = \frac{1}{\pi}\langle f(\theta),\cos(\theta_n)\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)\cos(\theta_n)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)\cos(n\theta)d\theta$$
$$= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi} f(\theta)\cos(n\theta)d\theta + \int_{\theta=\pi}^{\theta=2\pi} f(\theta)\cos(n\theta)d\theta\right] = \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi}\cos(n\theta)d\theta - \int_{\theta=\pi}^{\theta=2\pi}\cos(n\theta)d\theta\right]$$

$$b_n = \frac{1}{\pi}\langle f(\theta),\sin(\theta_n)\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)\sin(\theta_n)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)\sin(n\theta)d\theta$$
$$= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi} f(\theta)\sin(n\theta)d\theta + \int_{\theta=\pi}^{\theta=2\pi} f(\theta)\sin(n\theta)d\theta\right] = \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi}\sin(n\theta)d\theta - \int_{\theta=\pi}^{\theta=2\pi}\sin(n\theta)d\theta\right]$$