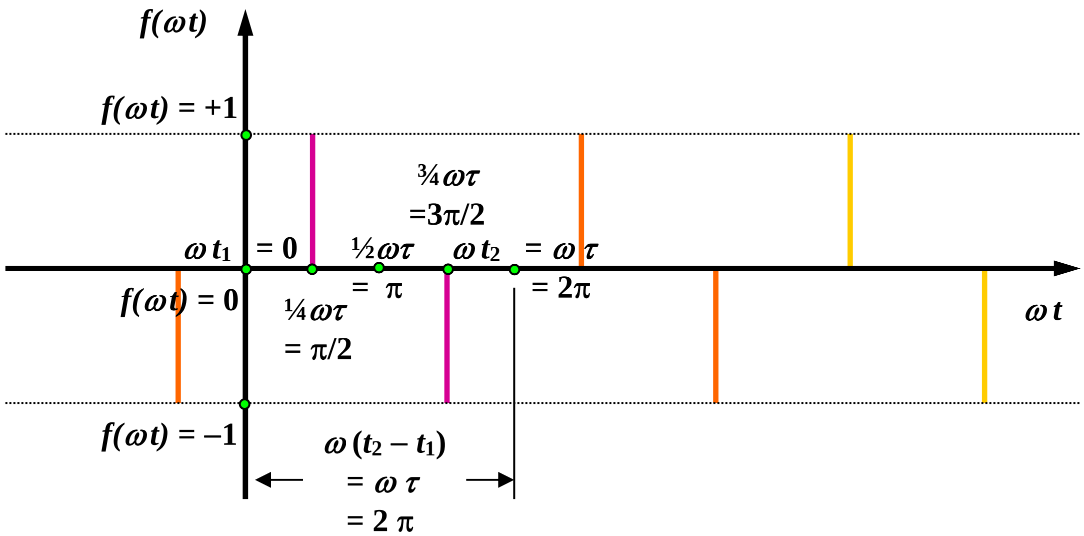
C. Fourier Analysis of a Periodic, Bipolar Delta-Function

The limiting case of the duty cycle going to zero for a temporally-periodic, bipolar square wave of is known as a periodic, bipolar *delta-function waveform*, consisting of a series of alternating up and down "spikes", each of zero width, as shown in the figure below:



Such "spikes" can be represented mathematically by a so-called *delta-function*, $\delta(x)$. The mathematical properties of the delta-function, $\delta(x)$ are quite intriguing. The delta-function, $\delta(x)$ is located at the position of its argument, here, $x = 0$. Thus, e.g. $\delta(x - x_o)$ is located at $x = x_o$, and thus $\delta(x + x_o)$ is located at $x = - x_o$. (n.b. The argument, u of the delta-function, $\delta(u)$ is *always* equal to zero, e.g. $u = (x - x_o) = 0$, thus $x = x_o$).

Formally, mathematically, the delta-function, $\delta(x)$ has *zero* width and *infinite* height, but *only* at $x = 0$. It is *zero* everywhere else. When a delta-function is used inside of an integral, amazing things happen as a result. For example, if the range of integration contains the point $x = x_o$, then:

$$\int \delta(x - x_o)dx = 1 \qquad\qquad \int f(x)\delta(x - x_o)dx = f(x_o)$$

otherwise both of these integrals are = 0, if $x = x_o$ is *not* contained within the range of integration. Note also that the (one-dimensional) delta-function, $\delta(x)$ has dimensions (i.e. units) of $1/x$.

Mathematically, we can define the above *odd-symmetry* waveform, $f(\theta)$ over the interval $0 \le \theta < 2\pi$ (i.e. one cycle of this waveform) as:

$$f(\theta) = \delta(\theta - \pi/2) - \delta(\theta - 3\pi/2)$$

Thus, the positive-going delta-function is located at $\theta = ¼\omega\tau = \pi/2$, and the negative-going delta-function is located at $\theta = ¾\omega\tau = 3\pi/2$.