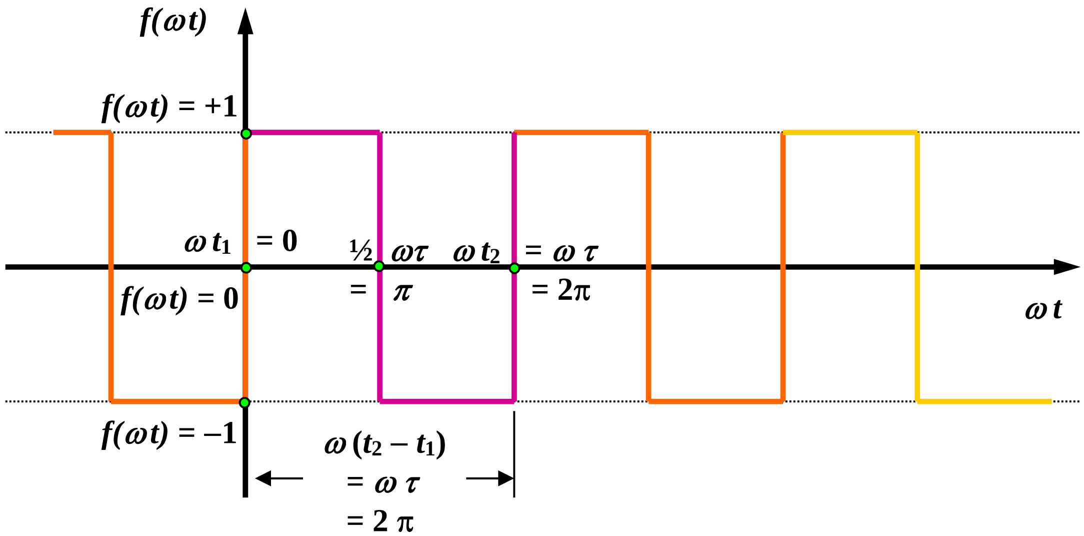
B. Fourier Analysis of a Periodic, Symmetrical Square Wave

A temporally-periodic, *bipolar* square wave of *unit* amplitude and 50% duty cycle is shown in the figure below:



Since this waveform repeats indefinitely, then, without any loss of generality we can *arbitrarily* choose (i.e. re-define) the starting time, t_1 of this waveform to be $t_1 = 0$ seconds. Thus the ending time, for one period of this waveform is $t_2 = \tau$ seconds. Then $\theta_1 = \omega t_1 = 0$, and $\theta_2 = \omega t_2 = \omega\tau = 2\pi f\tau = 2\pi/\tau * \tau = 2\pi$, since $f = 1/\tau$.

Mathematically, we define the square wave, for the one cycle as indicated in the figure above, as:

$$f(\theta) = f(\omega t) = +1 \quad \text{for} \quad 0 \le \theta < \pi$$

and:

$$f(\theta) = f(\omega t) = -1 \quad \text{for} \quad \pi \le \theta < 2\pi$$

This type of waveform is known as a *bipolar* square wave. It is positive, with unit amplitude for the first half of its cycle, and negative, with unit amplitude for the second half of its cycle. Thus, this waveform has a 50% duty cycle. Note also that this waveform has *odd* reflection symmetry, both about its θ-midpoint (i.e. its ωt-midpoint), $\theta = \omega t = ½\,\omega\tau = \pi$, *and* reflection about its $f(\theta) = f(\omega t) = 0$ midpoint. The waveform is said to have *odd* symmetry if it changes sign upon reflection, and has *even* symmetry if it does not change sign upon reflection.

Note also that this waveform is an example of a function which ***is*** *piece-wise continuous*. The waveform has *discrete*, but *finite* "jumps" when $\theta = 0$, π and 2π. Mathematically, the *slopes* of this waveform at these θ-points, i.e. the θ-derivatives of $f(\theta)$ are formally infinite:

$$\partial f(\theta)/\partial\theta|_{\theta=0} = +\infty, \quad \partial f(\theta)/\partial\theta|_{\theta=\pi} = -\infty, \quad \text{and} \quad \partial f(\theta)/\partial\theta|_{\theta=2\pi} = +\infty.$$

This is OK - mathematically we can deal with this just fine!