Similarly, *each* of the following inner products must vanish, for <u>*all*</u> values of *n* and *m*:

$$\int_{x=x_1}^{x=x_2} \frac{a_o}{2} * c_m \cos(mkx)dx = \frac{a_o c_m}{2} \int_{x=x_1}^{x=x_2} \cos(mkx)dx = +\frac{a_o c_m}{2}\left[\frac{\sin(mkx_2)}{mk} - \frac{\sin(mkx_1)}{mk}\right] = 0$$

$$\int_{x=x_1}^{x=x_2} \frac{a_o}{2} * d_m \sin(mkx)dx = \frac{a_o d_m}{2} \int_{x=x_1}^{x=x_2} \sin(mkx)dx = -\frac{a_o d_m}{2}\left[\frac{\cos(mkx_2)}{mk} - \frac{\cos(mkx_1)}{mk}\right] = 0$$

$$\int_{x=x_1}^{x=x_2} \frac{c_o}{2} * a_n \cos(nkx)dx = \frac{c_o a_n}{2} \int_{x=x_1}^{x=x_2} \cos(nkx)dx = +\frac{c_o a_n}{2}\left[\frac{\sin(nkx_2)}{nk} - \frac{\sin(nkx_1)}{nk}\right] = 0$$

$$\int_{x=x_1}^{x=x_2} \frac{d_o}{2} * b_n \sin(nkx)dx = \frac{d_o b_n}{2} \int_{x=x_1}^{x=x_2} \sin(nkx)dx = -\frac{d_o b_n}{2}\left[\frac{\cos(nkx_2)}{nk} - \frac{\cos(nkx_1)}{nk}\right] = 0$$

Each of these terms <u>*does*</u> vanish, because the functions *f(x)* and *g(x)* are periodic - i.e. they repeat themselves for $x_2 = x_1 + L$. Since the wavenumber, $k = 2\pi/L$, then for arbitrary values of *n*, *m* (= 1,2,3,...), then, e.g.:

$$\sin(mkx_2) = \sin(2\pi m x_2/L) = \sin(2\pi m(x_1+L)/L) = \sin(2\pi m x_1/L + 2\pi m) = \sin(2\pi m x_1/L)$$

$$\cos(mkx_2) = \cos(2\pi m x_2/L) = \cos(2\pi m(x_1+L)/L) = \cos(2\pi m x_1/L + 2\pi m) = \cos(2\pi m x_1/L)$$

These results explicitly demonstrate that, since the constant ($n = m = 0$) terms in the Fourier series, e.g. $a_o = a_o * 1$, that the *sin* (*mkx*) and *cos* (*mkx*) functions (with $m > 0$), as basis vectors, are orthogonal to 1 on the interval, $x_1 \le x \le x_2$.

Similarly, *each* of the following inner products must all vanish, for <u>*all*</u> values of *n* and *m*:

$$\int_{x=x_1}^{x=x_2} a_n c_m \cos(nkx)\cos(mkx)dx = +a_n c_m \left\{\left[\frac{\sin(n-m)kx_2}{2(n-m)k} + \frac{\sin(n+m)kx_2}{2(n+m)k}\right] - \left[\frac{\sin(n-m)kx_1}{2(n-m)k} + \frac{\sin(n+m)kx_1}{2(n+m)k}\right]\right\}$$

$$\int_{x=x_1}^{x=x_2} b_n c_m \sin(nkx)\cos(mkx)dx = -b_n c_m \left\{\left[\frac{\cos(n-m)kx_2}{2(n-m)k} - \frac{\cos(n+m)kx_2}{2(n+m)k}\right] - \left[\frac{\cos(n-m)kx_1}{2(n-m)k} - \frac{\cos(n+m)kx_1}{2(n+m)k}\right]\right\}$$

$$\int_{x=x_1}^{x=x_2} b_n d_m \sin(nkx)\sin(mkx)dx = +b_n d_m \left\{\left[\frac{\sin(n-m)kx_2}{2(n-m)k} - \frac{\sin(n+m)kx_2}{2(n+m)k}\right] - \left[\frac{\sin(n-m)kx_1}{2(n-m)k} - \frac{\sin(n+m)kx_1}{2(n+m)k}\right]\right\}$$

For the cases where $n \ne m$, each of the above three types of integrals *does* vanish, because the *sin* (*mkx*) and *cos* (*mkx*) functions are periodic on the interval, $x_1 \le x \le x_2$. These results explicitly demonstrate that for $n \ne m$, that the *cos* (*nkx*) and *cos* (*mkx*) functions, as basis vectors, are orthogonal to each other; the *sin* (*nkx*) and *cos* (*mkx*) functions are also orthogonal to each other; and the *sin* (*nkx*) and *sin* (*mkx*) functions are also orthogonal to each other on the interval, $x_1 \le x \le x_2$.