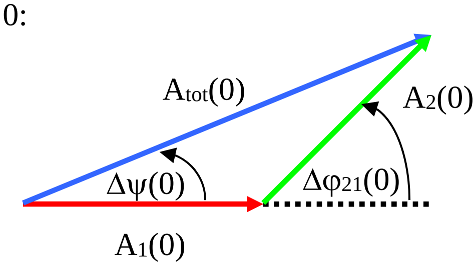
Beats Phenomenon

Linearly superpose (*i.e.* *add*) two signals with amplitudes $A_1(t)$ and $A_2(t)$, and which have similar/comparable frequencies, $\omega_2(t) \sim \omega_1(t)$, with instantaneous phase of the second signal relative to the first of $\Delta\varphi_{21}(t)$:

$$A_1(t) = A_{10}\cos\left(\omega_1(t)t\right) \qquad A_2(t) = A_{20}\cos\left(\omega_2(t)t + \Delta\phi_{21}(t)\right)$$
$$A_{tot}(t) = A_1(t) + A_2(t) = A_{10}\cos\left(\omega_1(t)t\right) + A_{20}\cos\left(\omega_2(t)t + \Delta\phi_{21}(t)\right)$$

Note that at the amplitude level, there is nothing explicitly overt and/or obvious in the above mathematical expression for the overall/total/resultant amplitude, $A_{tot}(t)$ that *easily* explains the phenomenon of beats associated with adding together two signals that have comparable amplitudes and frequencies.

However, let us consider the (instantaneous) phasor relationship between the individual amplitudes for the two signals, $A_1(t)$ and $A_2(t)$ respectively. Their relative initial phase difference at time $t = 0$ is $\Delta\varphi_{21}(t=0)$ and the resultant/total amplitude, $A_{tot}(t=0)$ is shown in the figure below, for time, $t = 0$:



From the law of cosines, the magnitude of the total amplitude, $A_{tot}(t)$ at an arbitrary time, t is obtained from the following:

$$A_{tot}^2(t) = A_1^2(t) + A_2^2(t) - 2A_1(t)A_2(t)\cos\left[\pi - \left((\omega_2(t) - \omega_1(t))t + \Delta\phi_{21}(t)\right)\right]$$

$$A_{tot}^2(t) = A_1^2(t) + A_2^2(t) + 2A_1(t)A_2(t)\cos\left((\omega_2(t) - \omega_1(t))t + \Delta\phi_{21}(t)\right)$$

Thus,

$$A_{tot}(t) = \sqrt{A_1^2(t) + A_2^2(t) + 2A_1(t)A_2(t)\cos\left((\omega_1(t) - \omega_2(t))t + \Delta\phi_{21}(t)\right)}$$
$$= \sqrt{\begin{array}{l} A_{10}^2\cos^2\left(\omega_1(t)t\right) + A_{20}^2\cos^2\left(\omega_2(t)t\right) \\ +2A_{10}A_{20}\cos\left(\omega_1(t)t\right)\cos\left(\omega_2(t)t + \Delta\phi_{21}(t)\right)\cos\left((\omega_2(t) - \omega_1(t))t + \Delta\phi_{21}(t)\right) \end{array}}$$

For equal amplitudes, $A_{10} = A_{20} = A_0$, zero relative initial phase, $\Delta\varphi_{21} = 0$ and constant (i.e. time-independent) frequencies, ω_2 and ω_1, this expression reduces to:

$$A_{tot}(t) = A_0\sqrt{\cos^2\omega_1 t + \cos^2\omega_2 t + 2\cos\omega_1 t\cos\omega_2 t\cos\left((\omega_2 - \omega_1)t\right)}$$

The phase of the total amplitude, $A_{tot}(t)$ relative to that of the first amplitude $A_1(t)$, at an arbitrary time, t is $\Delta\psi(t)$ and is obtained from the projections of the total amplitude phasor, $A_{tot}(t)$ onto the y- and x- axes of the 2-D phasor plane:

$$\tan(\Delta\psi) = \frac{A_2(t)\cos\left(\omega_2(t)t + \Delta\varphi_{21}(t)\right)\sin\Delta\varphi_{21}(t)}{A_1(t)\cos\left(\omega_1(t)t\right) + A_2(t)\cos\left(\omega_2(t)t + \Delta\varphi_{21}(t)\right)\cos\Delta\varphi_{21}(t)}$$