It can be readily seen from the above diagram that the endpoint of the complex "vector", Z = X+iY, lies at a point on the circumference of a circle, centered at (X,Y) = (0,0), with "radius" (i.e. magnitude) $|Z| = (X^2 + Y^2)^{1/2}$ and phase angle, φ (defined relative to the X-axis), of $\varphi \equiv \tan^{-1}(Y/X)$ (or equivalently: $\varphi' \equiv \tan^{-1}(X/Y)$, defined relative to the Y-axis).

Instead of using Cartesian coordinates, we can alternatively and equivalently express the complex variable, Z in polar coordinate form, $Z = |Z|(\mathrm{Cos}\varphi + i\mathrm{Sin}\varphi)$, since $X = |Z|\mathrm{Cos}\varphi$ and $Y = |Z|\mathrm{Sin}\varphi$. Recall also the trigonometric identity, $\mathrm{Cos}^2\varphi + \mathrm{Sin}^2\varphi = 1$, which is used in obtaining the magnitude of Z, |Z| from Z itself.

If we now redefine the variable φ such that $\varphi \Rightarrow (\omega t + \varphi)$, it can then be seen that $Z(t) = |Z|\{\mathrm{Cos}(\omega t + \varphi) + i\mathrm{Sin}(\omega t + \varphi)\}$, with real component $X(t) = \mathrm{Re}(Z(t)) = |Z|\mathrm{Cos}(\omega t + \varphi)$, and imaginary component $Y(t) = \mathrm{Im}(Z(t)) = |Z|\mathrm{Sin}(\omega t + \varphi)$. At time t = 0, these relations are identical to the above. If (for simplicity's sake) we take the phase angle, $\varphi = 0$, then $Z(t) = |Z|\{\mathrm{Cos}(\omega t) + i\mathrm{Sin}(\omega t)\}$. At time t = 0, it can be seen that the complex variable Z(t=0) is purely real, Z(t=0) = X(t=0), i.e. lying entirely along the x-axis, Z(t=0) = |Z|cos0 = |Z|. As time, t progresses, the complex variable $Z(t) = |Z|\{\mathrm{Cos}(\omega t) + i\mathrm{Sin}(\omega t)\}$ rotates in a counter-clockwise manner with (constant) angular frequency, $\omega = 2\pi f$ radians/second, where f is the frequency (in cycles/second {cps}, or Hertz {=Hz}), completing one revolution in the complex plane every $\tau = 1/f = 2\pi/\omega$ seconds. The variable τ is also known as the period of oscillation, or period of vibration.

Linear Superposition (Addition) of Two Periodic Signals

It is illustrative to consider the situation of linear superposition of two periodic, equal-amplitude, identical-frequency signals, but in which one signal differs in phase from the other by 90 degrees. Since the zero of time is arbitrary, we can thus chose one signal to be purely real at time t = 0, such that $Z_1(t) = A\mathrm{Cos}\omega t$ and the other signal, $Z_2(t) = iA\mathrm{Sin}\omega t$, with purely real amplitude, A and angular frequency, ω for each. Then, using the trigonometric identity Cos(A-B) = CosACosB + SinASinB, we see that $\mathrm{Sin}\omega t = \mathrm{Cos}(\omega t - 90^o) = \mathrm{Cos}\omega t\mathrm{Cos}90^o + \mathrm{Sin}\omega t\mathrm{Sin}90^o$. Thus, for this example, here, the signal $Z_2(t) = iA\mathrm{Sin}\omega t$ *lags* (i.e. is behind) the signal $Z_1(t) = A\mathrm{Cos}\omega t$ by 90 degrees in phase. The resultant/total complex amplitude, Z(t) is the sum of the two individual complex amplitudes, $Z(t) = Z_1(t) + Z_2(t) = A(\mathrm{Cos}\omega t + i\mathrm{Sin}\omega t)$. We can also write this relation in exponential form, since $\exp(i\varphi) = e^{i\varphi} \equiv (\mathrm{Cos}\varphi + i\mathrm{Sin}\varphi)$, $\exp(-i\varphi) = e^{-i\varphi} \equiv (\mathrm{Cos}\varphi - i\mathrm{Sin}\varphi)$, and thus $\mathrm{Cos}\varphi \equiv \frac{1}{2}(e^{i\varphi} + e^{-i\varphi})$ and $\mathrm{Sin}\varphi \equiv \frac{1}{2}i(e^{i\varphi} - e^{-i\varphi})$. Then for our example, the total complex amplitude, $Z(t) = Z_1(t) + Z_2(t) = A(\mathrm{Cos}\omega t + i\mathrm{Sin}\omega t) = Ae^{i\omega t}$, with magnitude $|Z(t)| = \sqrt{\{Z(t)Z^*(t)\}} = \sqrt{A^2} = A$.

If we had instead chosen the second amplitude to be $Z_2(t) = -Ai\mathrm{Sin}\omega t$, then the signal $Z_2(t)$ would *lead* (i.e. be ahead of) the signal $Z_1(t) = A\mathrm{Cos}\omega t$ by 90 degrees in phase. Then for this situation, the total complex amplitude, $Z(t) = Z_1(t) + Z_2(t) = A(\mathrm{Cos}\omega t - i\mathrm{Sin}\omega t) = Ae^{-i\omega t}$, with magnitude |Z(t)| = A (i.e. the same as before). Thus, a change in the sign of a complex quantity, $Z(t) \Rightarrow -Z(t)$ physically corresponds to a phase change/shift in phase/phase *advance* of +180 degrees (n.b. which is also mathematically equivalent to a phase *retardation* of –180 degrees).