The Left Null Space

Finally consider the **left null space** $N(A^T)$. Note that

$$A^T y = 0 \Leftrightarrow \left(A^T y\right)^T = y^T A = 0^T \Leftrightarrow \begin{bmatrix} y_1 & \cdots & y_m \end{bmatrix} A = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}$$

Since multiplying A on the left by a vector produces the zero vector, we call this the 'left' null space. How do we easily find such vectors y? One way is to note that this is just a null space, now for A^T rather than A, so use the null space info above to find a basis of special solutions.

An alternate approach is possible. Recall the factorization $A = LU$. This gives $L^{-1}A = U$. Each row of U is found by multiplying A on the left by a row of L^{-1} (think about this for a moment and you will see that it is true). So what if U has a zero row? Then the corresponding row of L^{-1} is in the left null space!

The rows of L^{-1} corresponding to zero rows of U are a basis of $N(A^T)$
$\dim N(A^T) = m - r = \text{nullity}(A^T)$