Characterizing the Column Space

Since the column space is the set of right-hand side vectors b giving consistency, we proceed to solve our system with a general b:

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 2 & 6 & 9 & 7 & b_2 \\ -1 & -3 & 3 & 4 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 3 & b_2 - 2b_1 \\ 0 & 0 & 6 & 6 & b_3 + b_1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 3 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & (b_3 + b_1) - 2(b_2 - 2b_1) \end{array}\right]$$

We conclude that

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ where } b_3 - 2b_2 + 5b_1 = 0 \left.\vphantom{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}\right\} \begin{array}{l} \text{restrictions} \\ \text{representation} \\ \text{of } C(A) \end{array}$$

$$b = u \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + v \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} + w \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} \left.\vphantom{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}\right\} \begin{array}{l} \text{linear combo} \\ \text{representation} \\ \text{of } C(A) \end{array}$$