More on Column and Null Spaces

Proof that $C(A)$ is a subspace: If b_1 and b_2 are in the column space, then each is a linear combo of the colums $a_1, ..., a_n$ of A: then $b_1 = x_1 a_1 + \cdots + x_n a_n$ and $b_2 = y_1 a_1 + \cdots + y_n a_n$ for some scalars $x_1, ..., x_2, y_1, ..., y_n$.Then

$$\begin{aligned} c_1 b_1 + c_2 b_2 &= c_1(x_1 a_1 + \cdots + x_n a_n) + c_2(y_1 a_1 + \cdots + y_n a_n) \\ &= (c_1 x_1 + c_2 y_1) a_1 + \cdots + (c_1 x_n + c_2 y_n) a_n \end{aligned}$$

which is again a linear combo of the columns of A.

Proof that $N(A)$ is a subspace: If x_1 and x_2 are in the null space, then $Ax_1 = 0$ and $Ax_2 = 0$. Therefore

$$A(c_1 x_1 + c_2 x_2) = c_1 A x_1 + c_2 A x_2 = c_1 0 + c_2 0 = 0$$

Hence $c_1 x_1 + c_2 x_2$ is also in the null space of A.