The PA = LU Factorization

Theorem: If A is non-singular, then there is a permutation matrix P such that $PA = LU$

i.e. G-E can be performed on PA

- A non-singular $\implies$ G-E with row exchanges works
- A singular $\implies$ G-E with row exchanges fails (you lose a pivot)

Example:

$$\begin{bmatrix} 0 & 0 & 2 \\ 1 & 3 & 3 \\ 1 & 4 & 5 \end{bmatrix} \to \underbrace{\begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 2 \\ 1 & 4 & 5 \end{bmatrix}}_{\text{Row 1} \leftrightarrow \text{Row 2}} \to \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \to \underbrace{\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}}_{\text{Row 2} \leftrightarrow \text{Row 3}}$$

so

$$P = [\text{Row 2} \leftrightarrow \text{Row 3}]\,[\text{Row 1} \leftrightarrow \text{Row 2}]$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$