SVD Example

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 \\ -4 & -4 \\ 2 & -2 \\ 3 & 1 \end{bmatrix}$$

and let us find its SVD, both as a matrix factorization and as a sum of rank 1 matrices. We begin with a computation:

$$A^T A = \begin{bmatrix} 1 & 3 \\ -4 & -4 \\ 2 & -2 \\ 3 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 3 \\ -4 & -4 \\ 2 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 30 & 18 \\ 18 & 30 \end{bmatrix}$$

It is now straight forward to find the eigenvalues and eigenvectors of $A^T A$ (and normalize the eigenvectors too):

$$\lambda_1 = 48, v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \lambda_2 = 12, v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

This means that so far we have:

$$\sigma_1 = \sqrt{48} = 2\sqrt{12}, \sigma_2 = \sqrt{12}, \Sigma = \begin{bmatrix} 2\sqrt[0]{12} & 0 \\ 0 & \sqrt{12} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

The first two columns of U can be computed directly:

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{2\sqrt{12}} \begin{bmatrix} 1 & 3 \\ -4 & -4 \\ 2 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{12}} \begin{bmatrix} 1 & 3 \\ -4 & -4 \\ 2 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

The last two vectors, u_3 and u_4 are an orthonormal basis of the nullspace of AA^T.