# Proof (continued)

Proof of (*): By direct calculation

$$AA^Tu_j = \frac{1}{\sigma_j}AA^TAv_j = \frac{1}{\sigma_j}A(\sigma_j^2 v_j) \;\text{ since } A^TAv_j = \sigma_j^2 v_j$$
$$= \sigma_j^2\left(\frac{1}{\sigma_j}Av_j\right) = \sigma_j^2 u_j$$

when $j = 1, ..., r$. So $u_j$ is an eigenvector of $AA^T$ corresponding to eigenvalue $\lambda_j = \sigma_j^2$. Also

$$(u_i, u_j) = u_i^Tu_j = \frac{1}{\sigma_i\sigma_j}(Av_i)^T(Av_j) = \frac{1}{\sigma_i\sigma_j}v_i^TA^TAv_j$$
$$= \frac{1}{\sigma_i\sigma_j}v_i^T(\sigma_j^2 v_j) = \frac{\sigma_j}{\sigma_i}v_i^Tv_j = \frac{\sigma_j}{\sigma_i}(v_i, v_j) \;\text{ since } A^TAv_j = \sigma_j^2 v_j$$

We see that when $i \neq j$, $(u_i, u_j) = 0$ because $(v_i, v_j) = 0$. And when $i = j$ we see $\|u_j\|^2 = \|v_j\|^2 = 1$. Thus $u_1, ..., u_r$ are orthonormal.