Proof of the Main Theorem

In summary we set

$$u_j = \frac{1}{\sigma_j} A v_j \quad , \; j = 1, \ldots, r$$

and $u_{r+1}, \ldots, u_m$ are any orthonormal basis of the null space of AA^T

Indeed, the proof of our theorem involves just proving that a) the u_j as computed this way are appropriate orthonormal eigenvectors of AA^T (equation (*)) and that $v_{r+1}, \ldots, v_n$ are in the null space of A (equation (**)).

Proof of (**): We have seen previously that A and $A^T A$ have the same nullspace. Another way to say this is that the eigenspaces of A and $A^T A$ corresponding to eigenvalue $\lambda = 0$ are the same. This proves (**).