More Precision in the Calculation of the SVF

There is some ambiguity in the preceding calculation (even though the one given here works). If x is an eigenvector that is a unit vector, so is $-x$. This can change the matrices U and V in such a way that our factorization doesn't work. More broadly, if the dimension of the eigenspace for some λ is larger than 1, Gram-Schmidt needs to be used to find an orthonormal basis and there are infinitely many orthonormal bases to chose from. There is, fortunately, a way to avoid this problem.

Write $A = U\Sigma V^T$ alternately as $AV = U\Sigma$. If $u_1, ..., u_m$ are the columns of U and $v_1, ..., v_n$ are the columns of V, then this last equation can be written alternately as

$$Av_j = \sigma_j u_j \quad , j = 1, ..., r \tag{*}$$

$$Av_j = 0 \quad , j = r + 1, ..., n \tag{**}$$

The first of these can be used to compute $u_1, ..., u_r$ and this removes the ambiguity.