More Observations

Moreover we note that

$$A^TA = \left(U\Sigma V^T\right)^T U\Sigma V^T = V\Sigma^T U^T U\Sigma V^T = V\left(\Sigma^T\Sigma\right)V^T$$

where $\Sigma^T\Sigma = \text{diag}(\sigma_1^2, ..., \sigma_r^2, \underbrace{0, ..., 0}_{n-r})$ is $n \times n$

But A^TA is ALWAYS a symmetric matrix and this is just its spectral factorization. We conclude that:

- the columns of V are an orthonormal basis of eigenvectors of A^TA, and
- $\sigma_1^2, ..., \sigma_r^2$ are the non-zero eigenvalues of A^TA, ordered from largest to smallest.

Ex: Consider

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$