The Singular Value Decomposition (SVD)

Inserting Σ we get

$$Av = U \left[\begin{array}{ccc|c} \sigma_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \sigma_r & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = U \begin{bmatrix} \sigma_1 x_1 \\ \vdots \\ \sigma_r x_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \sigma_1 x_1 u_1 + \cdots + \sigma_r x_r u_r = \sigma_1 (v_1^T v) u_1 + \cdots + \sigma_r (v_r^T v) u_r$$

$$= (\sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T) v$$

This gives us the **singular value decomposition** of A (SVD):

$$\boxed{A = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T}$$

It expresses A as a linear combination of rank one matrices weighted by the principal values of A.