A New Decomposition of Matrices

- $u_{r+1}, \ldots, u_m$ is an orthonormal basis of $N(A^T)$ (since $N(AA^T) = N(A^T)$)
- $v_1, \ldots, v_r$ is an orthonormal basis of $C(A^T)$
- $v_{r+1}, \ldots, u_n$ is an orthonormal basis of $N(A)$ (since $N(A^T A) = N(A)$)

Wow! Orthonormal bases everywhere!

Given any vector in $\mathbb{R}^n$ we can write

$$v = x_1 v_1 + \cdots + x_n v_n = V \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{where} \quad x_i = (v_i, v) = v_i^T v$$

Then

$$Av = (U\Sigma \underbrace{V^T) V}_{I} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = U\Sigma \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$