The Singular Value Factorization of a Matrix (SVF)

Theorem: Let A be an $m \times n$ matrix of rank r. Then A has the factorization

$$A = U \Sigma V^T$$

where U is an $m \times m$ orthogonal matrix, V is an $n \times n$ orthogonal matrix, and Σ is an $m \times n$ matrix that is "essentially" diagonal in the sense that it has the form

$$\Sigma = \underbrace{\left[\begin{array}{cccc|c} \sigma_1 & 0 & \cdots & 0 & 0 \\ 0 & \sigma_2 & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \sigma_r & 0 \\ \hline 0 & \cdots & \cdots & 0 & 0 \end{array}\right]}_{r \quad\quad n-r} \quad \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} r \\ \left.\vphantom{1}\right\} m-r \end{array}$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are the **singular values** of A. We call this the **singular value factorization** of A.