Spectral Decomposition

In this example we notice that the unit circle turns into an ellipse where the principal axes of the ellipse are the eigenvectors of A and the lengths of the semi-principal axes are the eigenvalues of A. (How does this picture change if one or both of A's eigenvalues are negative?)

There is a generalization of this example to n dimentions called the **Spectral Decomposition** of A. If v is any n-vector, we know

$$v = x_1 q_1 + \cdots + x_n q_n \quad \text{where} \quad x_i = (q_i, v) = q_i^T v$$

Therefore

$$\begin{aligned} Av &= x_1 A q_1 + \cdots + x_n A q_n = x_1 \lambda_1 q_1 + \cdots + x_n \lambda_n q_n \\ &= (q_1^T v)\lambda_1 q_1 + \cdots + (q_n^T v)\lambda_n q_n = (\lambda_1 q_1 q_1^T + \cdots + \lambda_n q_n q_n^T) v \end{aligned}$$