The Spectral Theorem:

Theorem: (Spectral factorization of a symmetric matrix)
Let A be a real symmetric matrix with eigenvalues $\lambda_1, ..., \lambda_n$. Then A has the factorization

$$A = Q\Lambda Q^T$$

where $\Lambda = \text{diag}(\lambda_1, ..., \lambda_n)$ and Q is an orthogonal matrix whose columns form an orthonormal basis of $\mathbb{R}^n$ **of eigenvectors of** A.

This is where orthonormal bases arise in a very natural way.

Sketch of the proof: By our previous calculations (since A is Hermitian), the eigenvalues of A are real numbers and

$$N(A - \lambda_i I) \perp N(A - \lambda_j I) \quad \text{for} \quad \lambda_i \neq \lambda_j$$

Using Gram-Schmidt we can construct from the eigenvectors for an eigenvalue λ an orthonormal basis of $N(A - \lambda I)$ for that same eigenvalue. Setting all these eigenvectors as columns of a matrix S, we get $A = S\Lambda S^{-1}$. But S is orthogonal, so call it Q and recall that $S^{-1} = Q^T$.