Proof (continued)

Step 2: The eigenvalues are real: if x and λ are such that $Ax = \lambda x$, then

$$\underbrace{x^H A x}_{\text{real}} = x^H(\lambda x) = \lambda x^H x = \lambda \underbrace{\|x\|^2}_{\text{real}}$$

and we see that λ is real.

Step 3: If $Ax = \lambda x$ and $Ay = \mu y$, $x \neq 0, y \neq 0, \lambda \neq \mu$, then $x \perp y$:

$$\begin{aligned}\mu(y,x) &= \mu y^H x = (\mu y)^H x \quad \text{(since } \mu \text{ is real)}\\ &= (Ay)^H x = y^H A^H x = y^H A x \quad \text{(since } A^H = A\text{)}\\ &= y^H(\lambda x) = \lambda(y,x)\end{aligned}$$

We see then that $(\mu - \lambda)(y,x) = 0$. But $\lambda \neq \mu$, and so $(y,x) = 0$.