The Fundamental Properties of Hermitian Matrices

Ex: $A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} \Rightarrow A^H = \bar{A}^T = \begin{bmatrix} 2 & 3+3i \\ 3-3i & 5 \end{bmatrix}^T = A$

Now we develop two fundamental properties of the eigenvalues and eigenvectors of Hermitian matrices.

Theorem: Suppose that $A^H = A$. Then all eigenvalues of A are real numbers. Moreover, if x and y are eigenvectors corresponding to different eigenvalues, then $x \perp y$.

Proof: Step 1: we show that $x^H Ax$ is real for any complex vector x.

For any x, $x^H Ax$ is a 1×1 matrix and hence equal to its own transpose:

$$x^H Ax = (x^H Ax)^T = (\bar{x}^T Ax)^T = x^T A^T \bar{x}$$

Therefore

$$\overline{x^H Ax} = \overline{x^T A^T \bar{x}} = \bar{x}^T \bar{A}^T x = x^H A^H x = x^H Ax$$

and any complex number that is its own conjugate is a real number.