Proof of the Main Theorem

Proof: Let the eigenvalues of A be λ_i with corresponding linearly independent eigenvectors x_i. From Part 1 of this topic we know that $u_i(t) = e^{\lambda_i t} x_i, i = 1, ..., n$ are n linearly independent solutions of $du/dt = Au$. By linearity we also know that

$$u(t) = c_1 u_1(t) + \cdots + c_n u_n(t) = c_1 e^{\lambda_1 t} x_1 + \cdots + c_n e^{\lambda_n t} x_n$$

is a solution for any constants c_i. This last expression can be rewritten (it is a linear combo) in matrix form as

$$\begin{aligned} u(t) &= \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix} \\ &= \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & 0 \\ 0 & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\ &= S e^{t\Lambda} c \end{aligned}$$