Example (continued)

From the eigenvalue information we know

$$A = S\Lambda S^{-1}, S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, S^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

and therefore

$$\begin{aligned} e^{tA} &= -\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} e^{-t}+e^{-3t} & e^{-t}-e^{-3t} \\ e^{-t}-e^{-3t} & e^{-t}+e^{-3t} \end{bmatrix} \end{aligned}$$

Now consider the linear differential equations problem

$$\begin{matrix} \frac{dv}{dt} = -2v + w, v(0) = 1 \\ \frac{dw}{dt} = v - 2w, w(0) = 0 \end{matrix} \iff \frac{du}{dt} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} u = Au, u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$