Matrix Exponentials via Diagonalization

The preceding calculation of $e^{\theta K}$ is a bit unusual in that for the majority of matrices a direct calculation such as was possible for it is not possible in general. But all is not lost. Diagonalization to the rescue!

Recall that if $A = S\Lambda S^{-1}$, then $A^k = S\Lambda^k S^{-1}$. This means that

$$\begin{aligned} e^A &= I + S\Lambda S^{-1} + \frac{1}{2!}S\Lambda^2 S^{-1} + \frac{1}{3!}S\Lambda^3 S^{-1} + \frac{1}{4!}S\Lambda^4 S^{-1} + \cdots \\ &= S(I + \Lambda + \frac{1}{2!}\Lambda^2 + \frac{1}{3!}\Lambda^3 + \frac{1}{4!}\Lambda^4 + \cdots)S^{-1} = Se^{\Lambda}S^{-1} \end{aligned}$$

And calculating the exponential of a diagonal matrix is much easier:

$$\begin{aligned} \Lambda &= \mathrm{diag}(\lambda_1, ..., \lambda_n) \Rightarrow \Lambda^k = \mathrm{diag}(\lambda_1^k, ..., \lambda_n^k) \\ &\Rightarrow e^{\Lambda} = \mathrm{diag}(1, ..., 1) + \mathrm{diag}(\lambda_1, ..., \lambda_n) + \frac{1}{2!}\mathrm{diag}(\lambda_1^2, ..., \lambda_n^2) + \cdots \\ &= \mathrm{diag}(1 + \lambda_1 + \frac{1}{2!}\lambda_1^2 + \cdots, ..., 1 + \lambda_n + \frac{1}{2!}\lambda_n^2 + \cdots) \\ &= \mathrm{diag}(e^{\lambda_1}, ..., e^{\lambda_n}) \end{aligned}$$